Math 5112 - Lecture #22

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Remark about quiver repns:

① a quiver repn $(X_\bullet, f_\bullet)$ is irreducible iff there is a unique vertex i such that $X_i = \mathbb{k}$, while $X_j = 0$ for $j \neq i$, and all maps $f_{jk} = 0$

② arbitrary quiver repns are not direct sums of irreducible repns $\Rightarrow$ hence, more useful + interesting to study indecomposable repns

Last time: Categories

A **category** consists of a class of objects, a class of morphisms between pairs of objects and a notion of composition for morphisms

↳ must be associative, have identity morphism

Useful terms: full subcategory, locally small,

↳ fewer objects, same morphisms ↳ Hom classes are sets

enriched (over another category D)

↳ Hom classes belong to D + some stuff about composition

There is a category of categories.

Objects are categories.

Morphisms are <u>functors</u> $F : C \to D$

A <u>functor</u> $F : C \to D$ assigns

① an object $F[x] \in D$ for each object $x \in C$

② a morphism $F[\sigma] : F[x] \to F[y]$ for each morphism $\sigma : x \to y$ in C

such that $F[id_x] = id_{F[x]}$ and $F[\sigma_1 \circ \sigma_2] = F[\sigma_1] \circ F[\sigma_2]$

ok, technically should say "category of small categories" or "2-category of categories" but let's not worry about this

There is also a category of functors $C \to D$

Objects are functors $C \to D$ and the morphisms are natural transformations $\alpha : F \to G$, which assigns to each object $X \in C$, a morphism $\alpha_X : F[X] \to G[X]$ in D such that

$$\begin{tikzcd} F[X] \arrow[r, "\alpha_X"] \arrow[d, "F[\sigma]"'] & G[X] \arrow[d, "G[\sigma]"] \\ F[Y] \arrow[r, "\alpha_Y"] & G[Y] \end{tikzcd}$$

always commutes.

In particular: functors can be composed, as can natural transformations

In any category $\mathcal{C}$, a morphism $f : X \to Y$ is an <u>isomorphism</u> if there is a morphism $f^{-1} : Y \to X$ such that $f \circ f^{-1} = id_Y$ and $f^{-1} \circ f = id_X$

<u>Ex</u> A <u>natural isomorphism</u> is a natural transformation that's an isomorphism in the functor category.

<u>Ex</u> Let $\text{FVect}_k = \left(\begin{array}{l} \text{finite-dim } k\text{-vector spaces} \\ \text{with linear maps as morphisms} \end{array} \right)$

There is a functor $** : \text{FVect}_k \to \text{FVect}_k$ and this is naturally isomorphic to the identity functor

Two categories C and D are equivalent if there exist functors $F: C \to D$ and $G: D \to C$ such that $F \circ G$ and $G \circ F$ are naturally isomorphic to identity functors. In this case, say that F and G are quasi-inverses.

Representable functors. Let C be a locally small category. A functor $F: C \to \mathrm{Sets}$ is representable if there is an object $X \in C$ such that $F \cong G$

naturally isomorphic

where $G = \mathrm{Hom}_C(X, \bullet) : C \to \mathrm{Set}$ is the functor

with $G[Y] = \mathrm{Hom}_C(X,Y) \in \mathrm{Sets}$

$$G[\sigma : Y \to Z] = \left(\begin{array}{ccc} \mathrm{Hom}_C(X,Y) & \longrightarrow & \mathrm{Hom}_C(X,Z) \\ \phi & \longmapsto & \sigma \circ \phi \end{array} \right)$$

In this case, we say that F is represented by X.

Yoneda lemma If $F : C \to \mathrm{Set}$ is represented by two objects X_1 and X_2 in C, then there is a unique isomorphism $\alpha : X_2 \to X_1$ such that the natural transformation $\mathrm{Hom}_C(X_1, \bullet) \to \mathrm{Hom}_C(X_2, \bullet)$ given by composition with α is a natural isomorphism.

Adjoint functors.

Two functors $F: C \to D$ and $G: D \to C$ are **adjoint** if for any $X \in C$, $Y \in D$, there is an isomorphism

if our categories are locally small, this just means a bijection

$$\varepsilon_{XY} : \mathrm{Hom}_D(F[X], Y) \xrightarrow{\sim} \mathrm{Hom}_C(X, G[Y])$$

such that

$$\begin{tikzcd}
\mathrm{Hom}_D(F[X_1], Y_1) \arrow[r, "\varepsilon_{X_1 Y_1}"] \arrow[d, "\phi \mapsto \beta \circ \phi \circ F[\alpha]"] & \mathrm{Hom}_C(X_1, G[Y_1]) \arrow[d, "\phi \mapsto G[\beta] \circ \phi \circ \alpha"] \\
\mathrm{Hom}_D(F[X_2], Y_2) \arrow[r, "\varepsilon_{X_2 Y_2}"] & \mathrm{Hom}_C(X_2, G[Y_2])
\end{tikzcd}$$

commutes for any morphisms $\alpha : X_2 \to X_1$ (in C) and $\beta : Y_1 \to Y_2$ (in D)

In this case, F is left adjoint to G and

G is right adjoint to F.

More concisely: F and G are adjoint functors if there is a natural isomorphism ε from the functor

$$\mathrm{Hom}_D(F[\bullet], \bullet) : C^{op} \times D \to \mathbf{Set}$$

to the functor

$$\mathrm{Hom}_C(\bullet, G[\bullet]) : C^{op} \times D \to \mathbf{Set}$$

<u>Ex</u> For finite groups $H \subset G$, Frobenius reciprocity implies that Res^G_H is left adjoint to Ind^G_H

as functors between $\mathrm{Rep}_k(H)$ and $\mathrm{Rep}_k(G)$

<u>Ex</u> The right adjoint of the functor $\text{Groups} \to k\text{-Algebras}$
$G \mapsto k[G]$

is the functor $GL_1 : k\text{-Algebras} \to \text{Groups}$
$A \mapsto GL_1(A) = A^\times$ group of units in A

Abelian categories

An **abelian category** is a category enriched over abelian groups that is equivalent to a full subcategory of the category of left A-modules over a ring A, that is closed under

- finite direct sums $\oplus$
- kernels (where $\ker(f) = \{x \mid f(x)=0\}$)
- cokernels (where $\operatorname{coker}(f: X \to Y) = Y / \operatorname{image}(f)$)
- and images

When working with an abelian category C, can always pretend that C is just the category of all left A-modules

Complexes and cohomology

Let A be an abelian category

A **complex** in A consists of a series of morphisms
indexed by $\mathbb{Z}$
in A

$$C_\bullet = \left(\cdots \xrightarrow{d_{i-1}} C_i \xrightarrow{d_i} C_{i+1} \xrightarrow{d_{i+1}} C_{i+2} \xrightarrow{d_{i+2}} \cdots \right)$$

object in A

Such that $d_{i+1} \circ d_i : C_i \to C_{i+2}$ is **zero** for all $i \in \mathbb{Z}$ (Since Hom-spaces are abelian groups, there is always a zero morphism)

The *cohomology* of a complex $C_\bullet$ is the sequence of abelian groups $H^i(C_\bullet) \overset{\text{def}}{=} \ker(d_i)/\text{image}(d_{i-1})$ for $i \in \mathbb{Z}$.

Note: in a complex, always have $\text{image}(d_{i-1}) \subset \ker(d_i)$

Complex $C_\bullet$ is *exact* if $H^i(C_\bullet) = 0 \ \forall i \in \mathbb{Z}$

We view any finite sequence of arrows $C_i \to C_{i+1} \to \dots \to C_j$ as the infinite sequence

$$\cdots \to C_i \xrightarrow{id} C_i \xrightarrow{id} C_i \xrightarrow{id} C_i \to C_{i+1} \to \cdots \to C_j \xrightarrow{id} C_j \xrightarrow{id} C_j \xrightarrow{id} C_j \xrightarrow{id} \cdots$$

This lets us talk about finite complexes and whether they're exact.

Complexes in A form a category where morphisms $f_\bullet : C_\bullet \to D_\bullet$ of complexes consist of commuting diagrams

$$\begin{tikzcd}
\cdots \arrow[r] & C_i \arrow[r] \arrow[d, "f_i"'] & C_{i+1} \arrow[r] \arrow[d, "f_{i+1}"'] & C_{i+2} \arrow[r] \arrow[d, "f_{i+2}"'] & \cdots \\
\cdots \arrow[r] & D_i \arrow[r] & D_{i+1} \arrow[r] & D_{i+2} \arrow[r] & \cdots
\end{tikzcd}$$

Terminology: (for a given complex)

i-cocycle $\overset{\text{def}}{=}$ element of $\ker(d_i)$

i-coboundary $\overset{\text{def}}{=}$ element of image(d_{i-1})

i^{th} cohomology classes $\overset{\text{def}}{=}$ the elements of $H^i(C_\bullet)$

A <u>short exact sequence</u> is an exact (finite) complex

$$(\cdots 0 \to 0 \to)\, 0 \to X \to Y \to Z \to 0 \,(\to 0 \to 0 \to \cdots)$$

Such a sequence is <u>split</u> if isomorphic as a complex to $0 \to X \to X \oplus Z \to Z \to 0$ for some objectr X, Z.