Math 5112 - Lecture # 23

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Last time: Let C and D be categories. → last time we assumed $D = \text{Sets}$

Assume C enriched over D → means $\text{Hom}_C(X,Y) \in D$

A functor $F : C \to D$ is **representable**

(and **represented** by some object $X \in C$) if

F is naturally isomorphic to the functor

$\text{Hom}_C(X, \bullet) : C \to D$

Note: $\text{Hom}_C(X, \bullet)$ is a functor $C \to D$

$\text{Hom}_C(\bullet, X)$ is a functor $C^{op} \to D$ ← Sometimes call this a "contravariant functor $C \to D$"

<u>Yoneda lemma</u> If $F: C \to D$ is represented by two objects x_1 and x_2 then $x_1 \cong x_2$,

moreover there is a unique isomorphism $x_1 \cong x_2$ that extends to a natural isomorphism $\mathrm{Hom}_C(x_2, \bullet) \cong \mathrm{Hom}_C(x_1, \bullet)$

<u>Adjoint functors</u> Let C and D be locally small cats.

with functors $F: C \to D$ and $G: D \to C$.

We say F is <u>left adjoint</u> to G or

G is <u>right adjoint</u> to F or

F and G are <u>adjoint</u> if...

the functors

$$\mathrm{Hom}_{\mathcal{D}}(F[\bullet], \bullet) : \mathcal{C}^{op} \times \mathcal{D} \to \mathrm{Set}$$

and

$$\mathrm{Hom}_{\mathcal{C}}(\bullet, G[\bullet]) : \mathcal{C}^{op} \times \mathcal{D} \to \mathrm{Set}$$

are naturally isomorphic.

(I've not defined product category $\mathcal{C}^{op} \times \mathcal{D}$ ...
this is what you would expect)

Abelian categories

A category C is <u>abelian</u> if

1. C is enriched over Abelian Groups
2. there is a zero object
3. for any morphism $f: A \to B$ in C, the objects $\ker(f)$, $\mathrm{image}(f)$, and $\mathrm{coker}(f)$ are all in C
4. Every injective / surjective morphism is "normal"

(technical condition)

general categorical definitions of ker, image, coker are more complicated than you would expect

More concrete definition: an abelian category is just a category equivalent to a full subcategory of left modules over some ring, closed under finite $\oplus$, images, kernels, and cokernels.

A complex in an abelian category C is a diagram

$$A_\bullet = \left(\cdots \xrightarrow{d_{i-1}} A_i \xrightarrow{d_i} A_{i+1} \xrightarrow{d_{i+1}} A_{i+2} \xrightarrow{d_{i+2}} \cdots \right)$$

where $d_{i+1} \circ d_i = 0 \quad \forall i \in \mathbb{Z}$

We view any finite diagram

$$A_i \to A_{i+1} \to \cdots \to A_j$$

as the infinite chain

$$\cdots A_i \xrightarrow{id} A_i \xrightarrow{id} A_i \xrightarrow{id} A_i \to A_{i+1} \to \cdots \to A_j \xrightarrow{id} A_j \xrightarrow{id} A_j \xrightarrow{id} A_j \xrightarrow{id} \cdots$$

Cohomology groups of $A_\bullet$ are quotients

$$H^n(A_\bullet) \overset{\text{def}}{=} \ker(d_n) / \operatorname{image}(d_{n-1})$$

$A_\bullet$ is <u>exact</u> if $H^n(A_\bullet) = 0 \;\; \forall n \in \mathbb{Z}$.

$A_\bullet$ is <u>short exact</u> if it has form

$$(\cdots \to 0 \to 0 \to)\, 0 \to X \to Y \to Z \to 0 \,(\to 0 \to 0 \to \cdots)$$

Complexes in C form their own category in which morphisms $f_\bullet : A_\bullet \to B_\bullet$ are commuting diagrams

$$\left(\begin{tikzcd}
\cdots \arrow[r] & A_i \arrow[r, "d_i^A"] \arrow[d, "f_i"'] & A_{i+1} \arrow[r] \arrow[d, "f_{i+1}"] & \cdots \\
\cdots \arrow[r] & B_i \arrow[r, "d_i^B"] & B_{i+1} \arrow[r] & \cdots
\end{tikzcd}\right)$$

Exact functors

Let C and D be abelian categories with a functor $F: C \to D$.

F is additive if $Hom_C(X, Y) \to Hom_D(F[X], F[Y])$

$$\sigma \longmapsto F[\sigma]$$

is an abelian group homomorphism.

In this case one can show that $F[X \oplus Y] \cong F[X] \oplus F[Y]$

Def The functor F is

- left exact if F is additive and $0 \to F[X] \to F[Y] \to F[Z]$ is exact for any exact sequence $0 \xrightarrow{0} X \xrightarrow{f} Y \xrightarrow{g} Z$
  ↳ means $0 = \ker f$, $\text{image } f = \ker g$

- <u>right exact</u> if F is additive and

$$F[X] \to F[Y] \to F[Z] \to 0$$

is exact whenever $X \to Y \to Z \to 0$ is exact.

- <u>exact</u> if F is both left- and right-exact

<u>Def</u> An abelian category is <u>semisimple</u> if any short exact sequence $0 \to X \to Y \to Z \to 0$ <u>splits</u> in the sense that it is isomorphic as a complex to

$$0 \xrightarrow{0} X \xrightarrow{x \mapsto (x,0)} X \oplus Z \xrightarrow{(x,z) \mapsto z} Z \xrightarrow{0} 0$$

$\underline{\text{Ex}}$ If char(k) does not divide $|G|$ for a finite group G, then $\text{Rep}_k(G)$ is semisimple.

To see this: consider an exact sequence of G-repns

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

Then $X \cong \text{image}(f) = \ker(g)$ = a subrepresentation of Y

Since every G-repn over k is completely reducible, we have
↳ direct sum of irreducibles

$Y / \ker(g) \cong Z$ and $Y \cong X \oplus Z$

Can show/check that between semisimple categories every additive functor is exact.

Ex (Left but not right exact)

Consider $\mathrm{Hom}_C(A,\bullet) : C \to$ Abelian groups.

This is left exact because if $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z$ is exact then we have another sequence

$$\underset{\mathrm{Hom}_C(A,0)}{\overset{\shortparallel}{0}} \to \mathrm{Hom}_C(A,X) \xrightarrow{①} \mathrm{Hom}_C(A,Y) \xrightarrow{②} \mathrm{Hom}_C(A,Z)$$

$\phi \longmapsto f\circ\phi \qquad \psi \longmapsto g\circ\psi$

This is exact as ① $f\circ\phi = 0$ iff $\phi = 0$ as f injective

② $g\circ f\circ\phi = 0$ as $g\circ f = 0$ and if $g\circ\psi = 0$ then

$g(\psi(a)) = 0 \ \forall a \in A \Rightarrow \psi(a) = f(x_a)$ for a unique $x_a \in X$ and can check that formula $\phi(a) \overset{\text{def}}{=} x_a$ is in $\mathrm{Hom}_{\mathcal{C}}(A, X)$ with $f \circ \phi = \psi$.

However, $\mathrm{Hom}_{\mathcal{C}}(A, \bullet)$ not always right exact.

Consider $\mathcal{C}$ = rings = $\mathbb{Z}$-modules and $A = \mathbb{Z}/2\mathbb{Z}$.

Apply this to $0 \to \mathbb{Z} \xrightarrow{n \mapsto 2n} \mathbb{Z} \xrightarrow{n \mapsto n \pmod 2} \mathbb{Z}/2\mathbb{Z} \to 0$

gives $0 \to \underbrace{\mathrm{Hom}(A, \mathbb{Z})}_{=0} \to \underbrace{\mathrm{Hom}(A, \mathbb{Z})}_{=0} \to \underbrace{\mathrm{Hom}(A, A)}_{\neq 0} \to 0$

But $0 \to (\text{nonzero}) \to 0$ is never exact.

Ex If A is an algebra and X is a right A-module then the functon $X \otimes_A \bullet : \begin{pmatrix} \text{left } A\text{-} \\ \text{modules} \end{pmatrix} \to \begin{pmatrix} \text{Abelian} \\ \text{groups} \end{pmatrix}$

is right but not left exact.

(Again take $X = \mathbb{Z}/2\mathbb{Z}$, $A = \mathbb{Z}$, and apply functor to $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$)

Projective modules

Let A be a k-algebra

Let P be a left A-module

Thm The following properties are equivalent:

① If $\alpha : M \to N$ is a surjective morphism of left A-modules and $\nu : P \to N$ is any morphism then $\exists$ morphism $\mu : P \to M$ such that

$$\begin{tikzcd} & P \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ M \arrow[rr, "\alpha"] & & N \end{tikzcd}$$

Commutes.

$\hookrightarrow \nu = \alpha \circ \mu$

② If $\alpha : M \to P$ is surjective morphism then $\exists$ morphism $\mu : P \to M$ such that $\alpha \circ \mu = id_P$

(say that α "splits")

③ There exists a left A-module Q such that $P \oplus Q$ is a free left A-module.

free means isomorphic to the module of maps $f : X \to A$ with $f^{-1}(A - \{0\})$ finite. for some set X.

④ $Hom_A(P, \bullet)$ is an exact functor A-mod $\to$ Abelian Groups.

Pf ① $\Rightarrow$ ② since ② is ① with $N = P$, $u = id$

② $\Rightarrow$ ③ since there is always a free module M and a surjective morphism $\alpha : M \to P$ and if this splits then $P \oplus \ker\alpha \cong M$

③ $\Rightarrow$ ④ because if P is free itself then $\mathrm{Hom}_A(P, \bullet)$ is always exact and if the direct sum of two complexes is exact then each summand must be exact.

Finally, ④ $\Rightarrow$ ① because if K is kernel of $\alpha : M \to N$ then the sequence

$$0 \to K \hookrightarrow M \xrightarrow{\alpha} N \to 0$$

is exact, so if applying $\mathrm{Hom}_A(P, \bullet)$ to this gives another exact sequence

$$0 \to \mathrm{Hom}_A(P,K) \to \mathrm{Hom}_A(P,M) \xrightarrow[\text{surjective}]{} \mathrm{Hom}_A(P,N) \to 0$$

$$\mu \longmapsto \alpha \circ \mu$$

then for any $\nu \in \mathrm{Hom}_A(P,N)$ there is $\mu \in \mathrm{Hom}_A(P,M)$ with $\alpha \circ \mu = \nu$ as desired. $\square$

We say that P is a projective (left A-)module when these equivalent properties hold.

<u>Def</u> A <u>projective resolution</u> of a left A-module M is an exact sequence

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0$$

such that P_i is a projective left A-module for all $i \geq 0$. <u>Ex</u>. Any module has a projective resolution (in which every P_i is a free module)

We define <u>projective right A-modules</u> and <u>projective resolutions</u> of right A-modules in same way, just replacing "left" by "right" as appropriate.

<u>Def</u> (Tor) Let M be a right A-module with projective resolution $P_\bullet$. Let N be a left A-module. For integers $i \geq 0$, define $\mathrm{Tor}_i^A(M,N) = \mathrm{Tor}_i(M,N)$ to be the $(-i)^{th}$ cohomology group of the complex

$$\begin{array}{ccccccc} i = & -2 & & -1 & & 0 & \\ (\cdots \to & P_2 \otimes_A N & \to & P_1 \otimes_A N & \to & P_0 \otimes_A N & \to 0) \end{array}$$

<u>Ex</u> $\mathrm{Tor}_0(M,N) = P_0 \otimes_A N \,/\, \ker(P_1 \otimes_A N \to P_0 \otimes_A N)$

$$\cong M \otimes_A N$$

HW exercise

Tor is the "derived functor" of the tensor product

<u>Def</u> (Ext) Let M be a left A-module with projective resolution $P_\bullet$, let N be another left A-module. For $i \geq 0$ define

$$\mathrm{Ext}^i_A(M,N) = \mathrm{Ext}^i(M,N)$$

to be the i^{th} cohomology group of

$$\begin{array}{ccccccc} i= & -1 & & 0 & & & \\ (& 0 & \to & \mathrm{Hom}_A(P_0,N) & \to & \mathrm{Hom}_A(P_1,N) & \to \cdots) \end{array}$$

<u>Ex</u> $\mathrm{Ext}^0(M,N) = \ker\left(\mathrm{Hom}_A(P_0,N) \to \mathrm{Hom}_A(P_1,N)\right)$

$$= \{\phi \in \mathrm{Hom}_A(P_0,N) \mid \ker\phi \supset \mathrm{image}(P_1 \to P_0)\}$$

$$\cong \mathrm{Hom}_A(M,N)$$

HW exercise

Ext is the "derived functor" of Hom

HW exercise: up to $\cong$ (in strong sense), neither $\mathrm{Ext}^i(M,N)$ nor $\mathrm{Tor}_i(M,N)$ depend on the choice of $P_\bullet$

① These definitions don't make it very clear how to compute anything (but this can be done)

② Difficult conjectures / thms in algebra / rep'n thry often have a way of being rephrased as concise statements involving Ext, Tor, etc.