Math 5112 - Lecture #24

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Last time: projective modules, projective resolutions
Ext, Tor

Let A be a k-algebra

<u>Def</u> A left/right A-module P is <u>projective</u> if the functor $\mathrm{Hom}_A(P, \bullet) : \begin{pmatrix} \text{left/right} \\ A\text{-modules} \end{pmatrix} \to \begin{matrix} \text{Abelian} \\ \text{Groups} \end{matrix}$

is exact. ← means preserves exactness of short exact sequences

Equivalent characterizations: Assume P is a left A-module

(there are similar statements for right A-modules)

Prop. P is projective iff there exists a unique morphism μ such that

$$\begin{tikzcd} & P \arrow[dl, "\mu"'] \arrow[dr] & \\ M \arrow[rr, "\text{surjective}"'] & & N \end{tikzcd}$$

commutes for any surjective morphism $M \to N$ and any morphism $P \to N$

Prop P is projective iff any surjective morphism $\alpha : M \to P$ splits in sense that $\alpha \circ \mu = id_P$ for some morphism $\mu : P \to M$.

Prop P is projective iff there exists a left A-module Q such that $P \oplus Q$ is a free A-module.

Def A projective resolution of a left/right A-module M is an exact sequence of left/right A-modules $\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0$

where each P_i is projective.

Fact Every left/right A-module has a projective resolution

Ex If M is projective then a projective resolution is

$$\cdots \to \underset{\cdots}{0} \to 0 \to \cdots \to \underset{P_1}{0} \to \underset{P_0}{M} \xrightarrow{id} M \to 0$$

Assume M is an A-module with projective resolution (left or right)

$$(*) \qquad \cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

Assume N is a left A-module

Def When M is a right A-module, we define

$$\mathrm{Tor}_i(M,N) = \ker(P_i \otimes_A N \to P_{i-1} \otimes_A N) \,/\, \mathrm{image}(P_{i+1} \otimes_A N \to P_i \otimes_A N)$$

a cohomology group of the complex obtained by tensoring $(*)$ with N

<u>Def</u> When M is a left A-module, we define

$$\mathrm{Ext}^i(M,N) = \frac{\ker\left(\mathrm{Hom}_A(P_i,N) \to \mathrm{Hom}_A(P_{i+1},N)\right)}{\mathrm{image}\left(\mathrm{Hom}_A(P_{i-1},N) \to \mathrm{Hom}_A(P_i,N)\right)}$$

also a cohomology of the complex formed by applying $\mathrm{Hom}_A(\bullet, N)$ to $(*)$, which reverses direction of all arrows.

The dual of "projective" is <u>injective</u>.

A left/right A-module I is <u>injective</u> if the functor $\mathrm{Hom}_A(\bullet, I) : \left(\begin{matrix}\text{left/right} \\ A\text{-modules}\end{matrix}\right)^{op} \to \text{Abelian Groups}$ is exact.

There are alternate characterizations, similar to projective modules.

Lifting idempotents

Let A be a ring and let $I \subset A$ be a 2-sided ideal

Assume I is nilpotent: there exists a number $k>0$ such that $a_1 a_2 \cdots a_k = 0$ for all $a_1, a_2, \ldots, a_k \in I$.

(In this case we write $I^k = 0$)

Fact If $a \in I$ then $1-a$ is a unit in $A^\times \subset A$.

Pf $(1-a)^{-1} = 1 + a + a^2 + a^3 + \cdots + a^k$ where $I^k = 0$. $\square$

Prop Suppose $e_0 \in A/I$ and $e_0^2 = e_0$.

1. There exists $e \in A$ such that $e^2 = e$ and
$$\underbrace{e + I}_{\text{image of } e \text{ under } A \to A/I} = e_0. \quad (\text{call } e \text{ a \underline{lift} of } e_0).$$

2. If $e' = (e')^2 \in A$ is any other lift of e_0 then
$$e' = (1-a)e(1-a)^{-1} \text{ for some } a \in I.$$

Pf Start by assuming $I^2 = 0$.

Let f be any elem of A with $f + I = e_0$.

Since $e_0^2 = f^2 + I = e_0 = f + I$

We have $f^2 - f \in I$. Let $a = f^2 - f \in I$.

We want to find $b \in I$ such that $e = f - b$

then it holds that $e^2 = e$. (Note: $e + I = f + I$ if $e - f \in I$)

To have $e^2 = e$, the equation b must satisfy is:

$$(f-b)^2 = f^2 - fb - bf \underbrace{+b^2}_{\substack{=0 \text{ since } I^2=0 \\ \text{and } b \in I}} = f - b$$

$$\Leftrightarrow \underbrace{f^2 - f}_{=a} = fb + bf - b \Leftrightarrow \boxed{a = fb + bf - b}$$

This equation holds for $b = (2f-1)\overset{=f^2-f}{a} \in I$.

$$fb + bf - b = \underbrace{(2f^2 - f)a}_{2a+f} + \underbrace{(2f^2 - f)a}_{2a+f} - (2f-1)a \quad (\text{since } af = fa)$$

$$= \cancel{2f}a + \underbrace{4a^2}_{\substack{=0 \text{ as } I^2=0 \\ \text{and } a\in I}} - \cancel{2fa} + a = a \checkmark$$

Thus $e \overset{\text{def}}{=} f-b = f-(2f-1)a$ is one lift of e_0.

Suppose $e' = (e')^2 \in A$ is another lift of e_0.

Then $e' = e+c$ for some $c \in I$.

1 $(e+c)^2 = e+c \iff \underbrace{\cancel{e^2}}_{=e} + ec + ce + \underbrace{\cancel{c^2}}_{=0} = \cancel{e}+c$

$$\iff \boxed{ec+ce = c}$$

2 $\boxed{ece = 0}$ as $ece + ce^2 = ece + ce = ce$

③ Thus $(1+\underbrace{ce-ec}_{\in I \text{ as } c\in I})\,e\,\underbrace{(1+ce-ec)^{-1}}_{=1-ce+ec \text{ since } I^2=0} = \dots$

$$\dots = (1+ce-ec)\,e\,(1-ce+ec)$$

$$= (e+ce)(1-ce+ec)$$

$$= (1+c)e(1-ce+ec)$$

$$= (1+c)(e+ec)$$

$$= e+\underbrace{ec}_{=c \text{ by } ①}+ce+\underbrace{\cancel{cec}}_{=0 \text{ as } \in I^2=0}$$

$$= e+c = e'$$

So e and e' are indeed conjugate by some element $1-a\in A^\times$ for $a\in I$.

This proves the result when $I^2 = 0$.

General case: assume by induction that there exists a lift of e_0 to $e_k = e_k^2 \in A/I^{k+1}$ that is unique up to conjugation by elements of $1 + I^k$. Then we can lift e_k to $e_{k+1} \in A/I^{k+2}$ since $(I^{k+1})^2 = 0$ in A/I^{k+1}. By induction conclude that we can lift e_0 to e_k for $k \gg 0$ such that $I^{k+1} = 0 \Rightarrow A/I^{k+1} = A$. $\square$

Def A <u>complete system of orthogonal idempotents</u> in an algebra B is a list of elements

$$e_1, e_2, \dots, e_n \in B$$

such that $1 = e_1 + e_2 + \dots + e_n$ and $e_i e_j = \begin{cases} 0 & \text{if } i \neq j \\ e_i & \text{if } i = j \end{cases}$

Cor If $a_1, a_2, \dots, a_m$ is a complete system of orthogonal idempotents in A/I then there is a complete system of orthogonal idempotents $e_1, e_2, \dots, e_m \in A$ such that $e_i + I = a_i$ for all i. (Here assume I is nilpotent)

Pf sketch If $m=2$ then lift a_1 to e_1 and set $e_2 = 1-e_1$. Then $e_1 e_2 = e_1 - e_1^2 = 0$ and

$$e_2^2 = 1 - 2e_1 + e_1^2 = 1 - e_1 = e_2 \text{ and } e_2 + I$$

$$= (1+I) - (e_1 + I) = 1_{A/I} - a_1 = a_2 \quad \checkmark$$

If $m > 2$ then lift a_1 to e_1 again, and let $e_2, e_3, \ldots, e_m$ be lifts of $a_i = (1-a_1)a_i(1-a_1)$ (for $i = 2, 3, \ldots, m$)

exist by induction

holds as $a_1 a_i = a_i a_1 = 0$

from $(1-a_1)A/I(1-a_1)$ to $(1-e_1)A(1-e_1)$ $\square$

unit of this algebra is $1-e_1$

everything element x of this algebra has $xe_1 = e_1 x = 0$

Below, whenever I say "module" it means "left A-module".

Let A be a finite dimensional k-algebra.

Let $M_1, M_2, \dots, M_n$ be a complete list of non-isomorphic irreducible A-modules

Thm 1 For each i there is unique / $\cong$ indecomposable finitely generated projective A-module P_i with $\dim \mathrm{Hom}_A(P_i, M_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

this implies that $P_i \not\cong P_j$ if $i \neq j$

② It holds that $A \cong \bigoplus_{i=1}^{n} \underbrace{(\dim M_i)\, P_i}_{= P_i^{\oplus \dim M_i}}$

③ Any indecomposable finitely generated projective A-module is isomorphic to P_i for a (unique) index i.

<u>Pf</u> Recall: $A/\mathrm{Rad}(A) = \bigoplus_{i=1}^{n} \overbrace{\mathrm{End}(M_i)}^{\cong \mathrm{Mat}_{d\times d}(\mathbb{k}) \text{ for } d = \dim M_i}$ and $\mathrm{Rad}(A)$ is a nilpotent 2-sided ideal.

Identify $A/\mathrm{Rad}(A)$ with a block diagonal matrix algebra

Now let $e_{ij}^{0} = \begin{bmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & & & \\ & & & & 1 & & \\ & & & & & 0 & \\ & & & & & & \ddots \\ & & & & & & & 0 \end{bmatrix} \in \mathrm{End}(M_i)$

position (j,j)

The elements e_{ij}^{0} $(1 \leq i \leq n$ and $1 \leq j \leq \dim M_i)$ form a complete system of orthogonal idempotents for $A/\mathrm{Rad}(A)$. Lift this to a system of idempotents $e_{ij} \in A$ and define $P_{ij} = Ae_{ij}$.

Then $A = \bigoplus_{1 \leq i \leq n} \bigoplus_{1 \leq j \leq \dim M_i} P_{ij}$

This implies that each P_{ij} is projective because A is a free A-module.

Also $e_{ij} M_k \overset{\text{exercise}}{\cong} \mathrm{Hom}_A(P_{ij}, M_k)$

$e_{ij} m \longmapsto (a e_{ij} \mapsto a e_{ij} m)$

So $\dim \mathrm{Hom}(P_{ij}, M_k) = \delta_{ik}$ since this is $\dim(e_{ij} M_k)$. Finally, P_{ij} is independent (up to $\cong$) of j as e_{ij} is conjugate to e_{ik}

by an element of $A^\times$.

Let $P_i = P_{i1} \cong P_{i2} \cong P_{i3} \cong \cdots$

We claim that P_i is indecomposable.

If $P_i = Q_1 \oplus Q_2$ then either

$$\mathrm{Hom}_A(Q_1, M_j) = 0 \ \forall j \quad \text{or}$$

$$\mathrm{Hom}_A(Q_2, M_j) = 0 \ \forall j$$

So either $Q_1 = 0$ or $Q_2 = 0$

So P_i is indecomposable.

Finally, every indecomposable fin. gen. projective A-module has to occur in decomp of A, so is $\cong P_i$ for some i. $\square$

<u>Def</u> The projective module P_i is the <u>projective cover</u> of M_i

Let $c_{ij} = \dim \mathrm{Hom}_A(P_i, P_j)$.

<u>Def</u> The matrix $C = [c_{ij}]_{1 \leq i,j \leq n}$

is called the <u>Cartan matrix</u> of A.