Math 5112 - Lecture #25

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Last time: let k be a field
let A be a finite dimensional k-algebra
[Below: module means left module]

Suppose $M_1, M_2, M_3, \ldots, M_n$ are a complete list of non-isomorphic irreducible A-modules.

Thm For each i, there is a unique-up-to-isomorphism indecomposable, finitely generated, projective A-module P_i such that $\dim \mathrm{Hom}_A(P_i, M_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

(Call P_i the projective cover of M_i.)

Moreover, it holds that $A \cong \bigoplus_{i=1}^{n} \underbrace{(\dim M_i) P_i}_{= P_i \oplus P_i \oplus \cdots \oplus P_i \ \dim M_i \text{ summands}}$

and $P_1, P_2, P_3, \ldots, P_n$ are a complete list of non-isomorphic, finitely generated indecomposable, projective A-modules.

<u>Rmk</u> If A is semisimple, then every A-module is completely reducible, and so any irreducible A-module is already projective, so we would have $P_i = M_i$ in theorem.

Dimension Let A be a ring and let M be a left A-module.

Def The **projective dimension** $pd(M)$ of M is the length of the shortest finite projective resolution $\cdots \to P_{d+1} \to P_d \to P_{d-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ where this resolution is said to be $\underbrace{\text{finite of length } d}$ if $P_d \neq 0$ and $P_{d+1} = P_{d+2} = P_{d+3} = \cdots = 0$.

If no finite projective resolution exists for M then we define $pd(M) = \infty$.

Ex If M is already projective then

$$\cdots \to 0 \to 0 \to \underset{\cdots}{\cdots} \to \underset{P_1}{0} \to \underset{P_0}{M} \to M \to 0$$

is a projective resolution of length zero, so $pd(M) = 0$.

Conversely, can only have $pd(M) = 0$ if M is projective.

Thm $pd(M) \leq d$ if and only if $Ext^i(M,N) = 0$ for all $i > d$ for every left A-module N.

Def The ring A has left (respectively, right) **homological dimension** d if every left (resp., right) A-module M has $pd(M) \leq d$ and equality holds for some choice of M.

If no such d exists then there is some A-module M with $pd(M) = \infty$, and in this case we say that A has infinite homological dimension.

Prop Homological dim of $K[x_1, x_2, \ldots, x_n]$ is n
(variables commute)

[As sketched in textbook, this can be shown using the Hilbert syzygies thm.]

Prop Homological dim of $k\langle x_1, x_2, \ldots, x_n\rangle$ is 1 (variables do not commute)

[This means that for every (left) module M over the free algebra, there is a short exact sequence

$$0 \to P_1 \to P_0 \to M \to 0 \quad \text{(with } P_i \text{ projective)}$$

but not every module M is projective.]

Blocks $\rightsquigarrow$ sometimes used in literature without a very standard definition.

all definitions will be variations on the following.

Let A be a finite-dimensional k-algebra.

Assume k is algebraically closed.

Define $\mathrm{Mod}_A \overset{\text{def}}{=}$ category of left A-modules

$\mathrm{FMod}_A \overset{\text{def}}{=}$ full subcategory of finite-dim. A-modules.

Def Two irreducible modules $X, Y \in \mathrm{FMod}_A$ are **linked** if there are modules $M_0, M_1, M_2, \dots, M_n \in \mathrm{FMod}_A$ such that $X \cong M_0$, $Y \cong M_n$, and for each $0 \leq i < n$, either $\mathrm{Ext}^1(M_i, M_{i+1}) \neq 0$ or $\mathrm{Ext}^1(M_{i+1}, M_i) \neq 0$.

[Note that if $X \cong Y$, then can take $n = 0$ to conclude that X, Y are linked]

Recall that a Jordan-Hölder series for $M \in \mathrm{Mod}_A$ is any sequence $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ such that each $M_i \in \mathrm{Mod}_A$ and each $Q_i \overset{\text{def}}{=} M_i / M_{i-1}$ is irreducible.

<u>Fact</u> Each $M \in \mathrm{FMod}_A$ has a Jordan-Hölder series and the quotient modules $Q_1, Q_2, \dots, Q_n$ are uniquely determined up to $\cong$ and permutation of indices.

Thus, well-defined to say "X appears in the Jordan-Hölder series of M" to mean that $X \cong Q_i$ for some i.

Def ① A module $M \in FMod\,A$ belongs to a **block** if all of the irreducible modules appearing in its Jordan-Hölder series are linked.

② Two modules $M, N \in FMod\,A$ belong to the same block if all irreducibles modules appearing in the Jordan-Hölder series for M are linked to those in N

Rmk $$Ext^1(M,N) \overset{def}{=} \frac{\ker(Hom_A(P_1,N) \to Hom_A(P_2,N))}{image(Hom_A(P_0,N) \to Hom_A(P_1,N))}$$

for some projective resolution $\cdots \to P_1 \to P_0 \to M \to 0.$

We saw a more concrete construction of $\mathrm{Ext}^1(M,N)$ for $M, N \in \mathrm{FMod}_A$, without referring to projective resolutions, in HW3:

$$\mathrm{Ext}^1(M,N) = \underbrace{Z^1(M,N)}_{\text{1-cocycles}} / \underbrace{B^1(M,N)}_{\text{1-coboundaries}}$$

$$\cong \left\{ \begin{array}{l} \text{vector space spanned by isomorphism} \\ \text{classes of } A\text{-modules } U \text{ that are} \\ \text{"nontrivial extensions" of } M, N \text{ in} \\ \text{sense that } M \subset U \text{ and } U/M \cong N \\ \text{and } U \not\cong M \oplus N \end{array} \right\}$$

For example, if A is semisimple, then every $M \in \mathrm{FMod}_A$ is a direct sum $M = \bigoplus_{i \in I} M_i^{\oplus n_i}$ where each M_i is irreducible, and so X appears in the Jordan-Hölder series for M iff $X \cong M_i$ where $n_i \neq 0$.

In this case $\mathrm{Ext}^1(M, N) = 0$ whenever M, N are irreducible, [If $M \subset U$ and $N \cong M/U$ then $U \cong M \oplus U/M \cong M \oplus N$] and so each block consists of all modules isomorphic to an element of $\{M, M \oplus M, M \oplus M \oplus M, \dots\}$ for some irreducible $M \in \mathrm{FMod}A$.

Prop. If A is semisimple then we have a bijection

$$\{\text{blocks}\} \longleftrightarrow \left\{\begin{array}{c}\text{isomorphism classes}\\ \text{of irreducible } A\text{-modules}\end{array}\right\}$$

Prop If M is indecomparable with Jordan-Hölder series $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ and $Q_i \overset{\text{def}}{=} M_i / M_{i-1}$ then $Q_1, Q_2, \dots, Q_n$ are all linked so M belongs to a block.

Pf If $n=1$ then M is irreducible, hence in a block
If $n=2$ then M is a nontrivial extension of $Q_1 = M_1$ and $Q_2 = M/Q_1$ since if $M \cong Q_1 \oplus Q_2$ then M would be decomparable, so $\mathrm{Ext}^1(Q_1, Q_2) \neq 0$ and result follows. If $n>2$, assume by induction that $Q_1, Q_2, \dots, Q_{n-1}$ are linked

and then apply $n=2$ case to M/Q_{n-2}. $\square$

Can also show:

<u>Prop</u> Each block contains a unique isomorphism class of indecomposables modules.

Finite abelian categories and Morita equivalence

Let C be an abelian category that is <u>k-linear</u> and of <u>finite length</u>, in sense that for each $A \in C$ there exists $0 = A_0 \subset A_1 \subset A_2 \subset \cdots \subset A_n = A$ where each $A_i \in C$, $n < \infty$, and A_i / A_{i-1} is irreducible.

Also assume that if $A \in C$ is irreducible then $\mathrm{Hom}_C(A, A) \cong K$.

<u>Def</u> If C has finitely many isomorphism classes of irreducible objects and "has enough projectives" (in sense that every object is a quotient of a projective object), then we say that C is a <u>finite abelian category</u>.

an object P such that $\mathrm{Hom}_C(P, \bullet)$ is exact functor

Def An object $P \in C$ is called a **projective generator** if P is projective and every object in C is a quotient of $P^{\oplus n}$ for some $n > 0$.

Ex If $C = \mathrm{Mod}_A$ then the free A-module A itself is a projective generator, when A is a finite-dim. algebra

Fact Assume C is finite abelian category. Then C has a projective generator.

Idea: take any projective object that contains the direct sum of all irreducible object / $\cong$ as a quotient.

Thm Any finite abelian category C is equivalent to the category of modules of some finite dimensional k-algebra B.

Explicitly, $C \cong \mathrm{Mod}_B$ where $B = \mathrm{End}(P)^{op}$ for any projective generator $P \in A$, via

$$\overset{\in C}{M} \longmapsto \mathrm{Hom}(P, M)^{\in \mathrm{FMod}\, B}$$

Def Two (finite-dimensional) k-algebras A and B are **Morita equivalent** if the categories $FMod_A$ and $FMod_B$ are equivalent.

Def A finite-dim. k-algebra B is **basic** if $B/Rad(B)$ is commutative.

Thm ① Any finite abelian category C is equivalent to $FMod_B$ for a basic algebra B that is unique up to $\cong$.

② Any finite-dim. algebra A is Morita equivalent to a unique (up to $\cong$) basic algebra B with $\dim B \leq \dim A$.