This document is intended as an **exact transcript** of the lecture, with extra summary and vocabulary sections for your convenience. By design, the material covered in lecture is exactly the same as what is in these notes. Due to time constraints, the notes may sometimes only contain limited illustrations, proofs, and examples; for a more thorough discussion of the course content, **consult the textbook**.

Summary

Quick summary of today's notes. Lecture starts on next page.

- If A and B are $n \times n$ matrices with $AB = I_n$ then $BA = I_n$ and $A^{-1} = B$.
- A *subspace* H of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ containing the zero vector that is closed under linear combinations. This means that $0 \in H$ and if $u, v \in H$ and $c \in \mathbb{R}$ then $u + v \in H$ and $cv \in H$.
- The *zero subspace* of $\mathbb{R}^n$ is the set $\{0\}$ with just the zero vector $0 \in \mathbb{R}^n$. Let A be an $m \times n$ matrix.

  The *column space* of A is the span of the columns of A. Denoted $\mathrm{Col}\, A$. This is a subspace of $\mathbb{R}^m$.

$$\mathrm{Col}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathbb{R}\text{-span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} a \\ b \\ a \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\} \subseteq \mathbb{R}^4$$

  The *null space* of A is the set of vectors $\mathrm{Nul}\, A = \{v \in \mathbb{R}^n : Av = 0\}$. This is a subspace of $\mathbb{R}^n$.

$$\mathrm{Nul}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x = y + 2z = 0 \right\} = \left\{ \begin{bmatrix} 0 \\ -2z \\ z \end{bmatrix} : z \in \mathbb{R} \right\} \subseteq \mathbb{R}^3.$$

- A *basis* for a subspace $H \subseteq \mathbb{R}^n$ is a linearly independent spanning set.

  The *standard basis* of $\mathbb{R}^n$ is $e_1, e_2, \dots, e_n$ where $e_i \in \mathbb{R}^n$ is the vector with 1 in row i and 0 in all other rows. Any subspace of $\mathbb{R}^n$ has a basis with at most n vectors.
- The pivot columns of an $m \times n$ matrix A form a basis for $\mathrm{Col}\, A$.
- Both A and $\mathsf{RREF}(A)$ have the same null space. Usually $\mathrm{Col}\, A \neq \mathrm{Col}\, \mathsf{RREF}(A)$.

  To find a basis for $\mathrm{Nul}\, A$, determine the indices $i_1, i_2, \dots, i_p$ of the non-pivot columns of A.

  Then there are unique vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$ such that any

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \qquad \text{with} \qquad \mathsf{RREF}(A)x = 0$$

  can be written as $x = x_{i_1} v_1 + x_{i_2} v_2 + \dots + x_{i_p} v_p$. The vectors $v_1, v_2, \dots, v_p$ are a basis for $\mathrm{Nul}\, A$.

  For example, if $\mathsf{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 4 & -1 \\ 0 & 0 & 1 & 0 & 2 \end{bmatrix}$ then any $x \in \mathbb{R}^5$ with $\mathsf{RREF}(A)x = 0$ has

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 4x_4 + x_5 \\ x_2 \\ -2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

  The three vectors on the right are a basis for $\mathrm{Nul}\, A = \mathrm{Nul}\, \mathsf{RREF}(A)$.

1 Last time: inverses

The following all mean the same thing for a function $f : X \to Y$:

1. f is *invertible*.
2. f is one-to-one and onto.
3. For each $b \in Y$ there is exactly one $a \in X$ with $f(a) = b$.
4. There is a unique function $f^{-1} : Y \to X$, called the *inverse* of f, such that

$$f^{-1}(f(a)) = a \qquad \text{and} \qquad f(f^{-1}(b)) = b \qquad \text{for all } a \in X \text{ and } b \in Y.$$

Proposition. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear and invertible then $m = n$ and T^{-1} is linear and invertible.

The following all mean the same thing for an $n \times n$ matrix A:

1. A is *invertible*.
2. A is the standard matrix of an invertible linear function $T : \mathbb{R}^n \to \mathbb{R}^n$.
3. There is a unique $n \times n$ matrix A^{-1}, called the *inverse* of A, such that

$$A^{-1}A = AA^{-1} = I_n \qquad \text{where we define } I_n = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

4. For each $b \in \mathbb{R}^n$ the equation $Ax = b$ has a unique solution.
5. $\mathsf{RREF}(A) = I_n$
6. The columns of A are linearly independent and their span is $\mathbb{R}^n$.

Proposition. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix.

(1) If $ad - bc = 0$ then A is not invertible.

(2) If $ad - bc \neq 0$ then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Proposition. Let A and B be $n \times n$ matrices.

1. If A is invertible then $(A^{-1})^{-1} = A$.
2. If A and B are both invertible then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
3. If A is invertible then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

Process to compute A^{-1}

Let A be an $n \times n$ matrix. Consider the $n \times 2n$ matrix $\begin{bmatrix} A & I_n \end{bmatrix}$.

If A is invertible then $\mathsf{RREF}\left(\begin{bmatrix} A & I_n \end{bmatrix}\right) = \begin{bmatrix} I_n & A^{-1} \end{bmatrix}$.

So to compute A^{-1}, row reduce $\begin{bmatrix} A & I_n \end{bmatrix}$ to reduced echelon form, and then take the last n columns.

2 Stronger characterization of invertible matrices

Remember that a matrix can only be invertible if it has the same number of rows and columns.

Theorem. When A is an $n \times n$ matrix, the following are equivalent:

(a) A is invertible.

(b) The columns of A are linearly independent.

(c) The span of the columns of A is $\mathbb{R}^n$

Proof. We already know that (a) implies both (b) and (c).

Assume just (b) holds. Then A has a pivot position in every column, so $\mathsf{RREF}(A) = I_n$ since A has the same number of rows and columns. But this implies that A is invertible.

Similarly, if (c) holds then A has a pivot position in every row, so $\mathsf{RREF}(A) = I_n$ and A is invertible. □

Corollary. Suppose A and B are both $n \times n$ matrices. If $AB = I_n$ then $BA = I_n$.

This means that if we want to show that $B = A^{-1}$ then it is enough to just check that $AB = I_n$.

Proof. Assume $AB = I_n$. Then the columns of A span $\mathbb{R}^n$ since if $v \in \mathbb{R}^n$ then $Au = v$ for $u = Bv \in \mathbb{R}^n$, so A is invertible. Therefore $B = A^{-1}AB = A^{-1}I_n = A^{-1}$ so $BA = A^{-1}A = I_n$. □

Important note: this corollary only applies to *square matrices*.

3 Subspaces of $\mathbb{R}^n$

Let n be a positive integer. Write $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$.

Definition. Let H be a subset of $\mathbb{R}^n$. The subset H is a *subspace* if these three conditions hold:

1. $0 \in H$.
2. $u + v \in H$ for all $u, v \in H$.
3. $cv \in H$ for all $c \in \mathbb{R}$ and $v \in H$.

Common examples

$\mathbb{R}^n$ is a subspace of itself.

The set $\{0\}$ consisting of just the zero vector is a subspace of $\mathbb{R}^n$.

The empty set $\varnothing$ is *not* a subspace since it does not contain the zero vector.

A subset $H \subseteq \mathbb{R}^2$ is a subspace if and only if $H = \{0\}$ or $H = \mathbb{R}^2$ or $H = \mathbb{R}\text{-span}\{v\}$ for some $v \in \mathbb{R}^2$

The span of a set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

Conversely, any subspace of $\mathbb{R}^n$ is the span of a finite set of vectors, although this is not obvious.

Example. The set

$$X = \left\{ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 1 \right\}$$

is *not* a subspace since $0 \notin X$.

Example. The set

$$H = \left\{ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 0 \right\}$$

is a subspace since if $u, v \in H$ and $c \in \mathbb{R}$ then

$$(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) = (u_1 + u_2 + u_3) + (v_1 + v_2 + v_3) = 0 + 0 = 0$$

and

$$cv_1 + cv_2 + cv_3 = c(v_1 + v_2 + v_3) = 0$$

so $u + v \in H$ and $cv \in H$.

Any matrix A gives rise to two subspaces, called the *column space* and *null space*.

Definition. The *column space* of an $m \times n$ matrix A is the subspace

$$\mathrm{Col}\, A \subseteq \mathbb{R}^m$$

given by the span of the columns of A.

Example. If $V = \mathbb{R}\text{-span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ then what are some matrices A with $\mathrm{Col}\, A = V$?

Here are four examples:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}.$$

Many different matrices can have the same column space, and it may not be at all obvious whether a subspace V is equal to the column space of a given matrix A.

Remark. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is the linear function $T(x) = Ax$ then $\mathrm{Col}\, A = \mathsf{range}(T)$.

A vector $b \in \mathbb{R}^m$ belongs to $\mathrm{Col}\, A$ if and and only if $Ax = b$ has a solution.

Therefore $\mathrm{Col}\, A = \mathbb{R}^m$ if and only if $Ax = b$ has a solution for each $b \in \mathbb{R}^m$.

Definition. The *null space* of an $m \times n$ matrix A is the subspace

$$\mathrm{Nul}\, A \subseteq \mathbb{R}^n$$

given by the set of vectors $v \in \mathbb{R}^n$ with $Av = 0$.

Proof that $\mathrm{Nul}\, A$ *is a subspace.* If $u, v \in \mathrm{Nul}\, A$ and $c \in \mathbb{R}$ then $A(u + v) = Au + Av = 0 + 0 = 0$ and $A(cv) = c(Av) = 0$, so $u + v \in \mathrm{Nul}\, A$ and $cv \in \mathrm{Nul}\, A$. Thus $\mathrm{Nul}\, A$ is a subspace of $\mathbb{R}^n$. □

Remark. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is the linear function $T(x) = Ax$ then $\operatorname{Nul} A = \{x \in \mathbb{R}^n : T(x) = 0\}$.

The column space is a subspace of $\mathbb{R}^m$ **where** m **is the number of rows of** A.

The null space is a subspace of $\mathbb{R}^n$ **where** n **is the number of columns of** A.

A subspace can be completely determined by a finite amount of data. This data will be called a *basis*.

Definition. Let H be a subspace of $\mathbb{R}^n$. A *basis* for H is a set of vectors $\{v_1, v_2, \dots, v_k\} \subseteq H$ that are linearly independent and have span equal to H.

The empty set $\varnothing = \{\}$ is considered to be a basis for the zero subspace $\{0\}$.

Example. The set $\{e_1, e_2, \dots, e_n\} \subseteq \mathbb{R}^n$ where $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, and so on, is a basis for $\mathbb{R}^n$.

We call this the *standard basis* of $\mathbb{R}^n$.

Theorem. Every subspace H of $\mathbb{R}^n$ has a basis of size at most n.

Proof. If $H = \{0\}$ then $\varnothing$ is a basis.

Assume $H \neq \{0\}$. Let $\mathcal{B}$ be a set of linearly independent vectors in H that is as large as possible. The size of $\mathcal{B}$ must be at most n since any $n+1$ vectors in $\mathbb{R}^n$ are linearly dependent.

Let $w_1, w_2, \dots, w_k$ be the elements of $\mathcal{B}$. Since $\mathcal{B}$ is as large as possible, if $v \in H$ is any vector then $w_1, w_2, \dots, w_k, v$ are linearly dependent so we can write

$$c_1 w_1 + c_2 w_2 + \dots + c_k w_k + cv = 0$$

for some numbers $c_1, c_2, \dots, c_k, c \in \mathbb{R}$ which are not all zero.

If $c = 0$ then this would imply that the vectors in $\mathcal{B}$ are linearly dependent. But the vectors in $\mathcal{B}$ are linearly independent, so we must have $c \neq 0$. Therefore

$$v = \tfrac{c_1}{c} w_1 + \tfrac{c_2}{c} w_2 + \dots + \tfrac{c_k}{c} w_k.$$

This means that v is in the span of the vectors in $\mathcal{B}$. Since $v \in H$ is an arbitrary vector, we conclude that the span of the vectors in $\mathcal{B}$ is all of H, so $\mathcal{B}$ is a basis for H. □

Example. Let $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$.

How can we find a basis for $\operatorname{Nul} A$? Well, finding a basis for $\operatorname{Nul} A$ is more or less the same task as finding all solutions to the homogeneous equation $Ax = 0$. So let's first try to solve that equation.

If we row reduce the 3×6 matrix $\begin{bmatrix} A & 0 \end{bmatrix}$, we get

$$\begin{bmatrix} A & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(\begin{bmatrix} A & 0 \end{bmatrix}).$$

This tells us that $Ax = 0$ if and only if $\begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \end{cases}$ or equivalently $\begin{cases} x_1 = 2x_2 + x_4 - 3x_5 \\ x_3 = -2x_4 + 2x_5. \end{cases}$

By substituting these formulas for the basic variables x_1 and x_3, we deduce that $x \in \operatorname{Nul} A$ if and only if

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

are a basis for $\operatorname{Nul} A$: we just computed that these vectors span the null space, and they are linearly independent since each has a nonzero entry in a row (namely, either row 2, 4, or 5) where the others have zeros. (Why does this imply linear independence?)

This example is important: the procedure just described works to construct a basis of $\operatorname{Nul} A$ for any matrix A. **The size of this basis will always be equal to the number of free variables in the linear system $Ax = 0$.** How to find a basis for $\operatorname{Nul} A$ is something you should remember at the end of this course.

Example. Let $B = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

This matrix is in reduced echelon form. How to find a basis for $\operatorname{Col} B$?

The columns of B automatically span $\operatorname{Col} B$, but they might not be linearly independent.

The largest linearly independent subset of the columns of B will be a basis for $\operatorname{Col} B$, however.

In our example, the pivot columns 1, 2 and 5 are linearly independent since each has a row with a 1 where the others have 0s. These columns span columns 3 and 4, so a basis for $\operatorname{Col} B$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

This example was special since the matrix B was already in reduced echelon form. To find a basis of the column space of an arbitrary matrix, we rely on the following observation:

Proposition. Let A be any matrix. The pivot columns of A form a basis for $\operatorname{Col} A$.

Proof. Let $v_1, v_2, \dots, v_n$ be the columns of $A = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$.

Consider the matrices $A_k = \begin{bmatrix} v_1 & v_2 & \dots & v_k \end{bmatrix}$ for $k = 1, 2, \dots, n$.

Observe that $\mathsf{RREF}(A_k)$ is equal to **the first k columns** of $\mathsf{RREF}(A)$.

If k is not a pivot column of A, then the last column of A_k is not a pivot column.

This means that $A_{k-1}x = v_k$ is consistent so v_k is in the span of $v_1, v_2, \dots, v_{k-1}$.

Thus each non-pivot column of A is a linear combination of earlier columns. This means that each non-pivot column of A is a linear combination of earlier columns **that are pivot columns**: if i_1 is the first non-pivot column, then v_{i_1} is a linear combination of earlier columns, which are all pivots; if i_2 is the second non-pivot column, then v_{i_2} is a linear combination of earlier columns, and these are all either pivots or v_{i_1}, but in any linear combination involving v_{i_1} we can replace v_{i_1} by a linear combination of pivot columns to get a linear combination involving only pivot columns; if i_3 is the third non-pivot column, then v_{i_3} is a linear combination of earlier columns, and these are all either pivots or v_{i_1} or v_{i_2}, and we can replace v_{i_1} and v_{i_2} by combinations of pivot columns as needed; and so on.

We conclude that $\mathrm{Col}\, A$ **is spanned by the pivot columns of** A. Why are they linearly independent?

If k is a pivot column of A, then the last column of A_k is a pivot column.

This means that $A_{k-1}x = v_k$ **is inconsistent so** v_k **is not in the span of** $v_1, v_2, \dots, v_{k-1}$**.**

Therefore v_k is also not in the span of the (smaller) set of earlier columns **that are pivot columns**.

Thus if $j_1 < j_2 < \cdots < j_q$ are the pivot columns of A then we have a strictly increasing chain of subspaces

$$\mathbb{R}\text{-span}\{v_{j_1}\} \subsetneq \mathbb{R}\text{-span}\{v_{j_1}, v_{j_2}\} \subsetneq \mathbb{R}\text{-span}\{v_{j_1}, v_{j_2}, v_{j_3}\} \subsetneq \cdots \subsetneq \mathbb{R}\text{-span}\{v_{j_1}, v_{j_2}, \dots, v_{j_q}\}.$$

The fact that this chain is strictly increasing means $v_{j_1}, v_{j_2}, \dots, v_{j_q}$ are **also linearly independent**.

(See Lecture 5 for the explanation of why this property is equivalent to linear independence.) $\square$

Example. The matrix

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix}$$

is row equivalent to the matrix B in the previous example. Columns 1, 2, and 5 of A have pivots, so

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -9 \\ 2 \\ 1 \\ -8 \end{bmatrix} \right\}$$

is a basis for $\mathrm{Col}\, A$.

Next time: we will show that if H is a subspace of $\mathbb{R}^n$ then all of its bases have the same size. The common size of each basis is the *dimension* of H.

4 Vocabulary

Keywords from today's lecture:

1. **Subspace** of $\mathbb{R}^n$

   A subset $H \subseteq \mathbb{R}^n$ such that $0 \in H$; if $u, v \in H$ then $u + v \in H$; and if $v \in H$, $c \in \mathbb{R}$ then $cv \in H$.

   Example: Pick any vectors $v_1, v_2, \dots, v_p \in \mathbb{R}^n$. Then $\mathbb{R}\text{-span}\{v_1, v_2, \dots, v_p\}$ is a subspace.

2. **Column space** of an $m \times n$ matrix A.

   The subspace $\mathrm{Col}\, A = \{Av : v \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$. The span of the columns of A.

   Example: If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ then $\mathrm{Col}\, A = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \in \mathbb{R}^3 : x, y \in \mathbb{R} \right\}$.

3. **Null space** of an $m \times n$ matrix A.

   The subspace $\mathrm{Nul}\, A = \{v \in \mathbb{R}^n : Av = 0\} \subseteq \mathbb{R}^n$.

   Example: If $A = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 2 & 0 \end{bmatrix}$ then $\mathrm{Nul}\, A = \left\{ \begin{bmatrix} 2x \\ x \\ y \end{bmatrix} \in \mathbb{R}^3 : x, y \in \mathbb{R} \right\} = \mathbb{R}\text{-span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

4. **Basis** of a subspace $H \subseteq \mathbb{R}^n$

   A set of linearly independent vectors in H whose span is H.

   Example: The vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ are a basis for the subspace $\left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 0 \right\}$.

   The **standard basis** of $\mathbb{R}^n$ consists of the vectors $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\dots$, $e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$.