This document is intended as an **exact transcript** of the lecture, with extra summary and vocabulary sections for your convenience. By design, the material covered in lecture is exactly the same as what is in these notes. Due to time constraints, the notes may sometimes only contain limited illustrations, proofs, and examples; for a more thorough discussion of the course content, **consult the textbook**.

Summary

Quick summary of today's notes. Lecture starts on next page.

- Let H be a subspace of $\mathbb{R}^n$.

  Every basis of H has the same size.

  The size of any basis of H is called its *dimension*. This number is denoted $\dim H$.

  We always have $0 \leq \dim H \leq n$.

  If $\dim H = d$ then we say that H is *d-dimensional*.

  Dimension measures the size of a subspace.

  We usually do not think of individual vectors as having dimension, since a single vector belongs to many different subspaces at the same time, all with different dimensions.

- Only the zero subspace has dimension 0.

  The only subspace of $\mathbb{R}^n$ with dimension n is $\mathbb{R}^n$ itself.

  If $U \subseteq V \subseteq \mathbb{R}^n$ are subspaces then $0 \leq \dim U \leq \dim V \leq n$.

- If $\mathcal{B} = (v_1, v_2, \dots, v_m)$ is a basis for a subspace H of $\mathbb{R}^n$, then each $h \in H$ can be expressed as

$$h = \begin{bmatrix} v_1 & v_2 & \cdots & v_m \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \quad \text{for a unique vector} \quad \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix} \in \mathbb{R}^m.$$

  The vector on the right is the *coordinate vector* of h in the basis $\mathcal{B}$, sometimes denoted $[h]_\mathcal{B} \in \mathbb{R}^m$.

- Let A be an $m \times n$ matrix.

  The dimension of $\operatorname{Col} A$ is the number of pivot columns in A.

  The dimension of $\operatorname{Nul} A$ is the number of non-pivot columns in A.

  Consequently $\dim \operatorname{Col} A + \dim \operatorname{Nul} A = n =$ the total number of columns in A.

- The *rank* of A is defined to be $\operatorname{rank} A = \dim \operatorname{Col} A$.

  A is invertible if and only if $\operatorname{rank} A = m = n$.

  Assume $m = n$. Then A is invertible if and only if $\operatorname{Nul} A = \{0\}$.

- Suppose H of $\mathbb{R}^n$ is a subspace and $p = \dim H$.

  Any set of p linearly independent vectors in H is a basis for H.

  Any set of p vectors whose span in H is a basis for H.

1 Last time: inverses and subspaces

To show that an $n \times n$ matrix A is *invertible*, all we have to do is check that (1) its columns are linearly independent or (2) its columns span $\mathbb{R}^n$. If either (1) or (2) holds, then the other property is also true.

If A is invertible then it has a unique *inverse* which is an $n \times n$ matrix A^{-1} with

$$AA^{-1} = A^{-1}A = I_n = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

If A and B are $n \times n$ and $AB = I_n$ then it automatically holds that $BA = I_n$ so $B = A^{-1}$ and $A = B^{-1}$.

(If A and B are not both $n \times n$ then it is possible to have $AB = I_n$ but $BA \neq I_n$.)

Definition. A subset H of $\mathbb{R}^n$ is a *subspace* if $0 \in H$, $u + v \in H$, and $cv \in H$ for all $u, v \in H$ and $c \in \mathbb{R}$.

A subspace is a nonempty set that contains all linear combinations of vectors in the set.

Example. Examples of subspaces of $\mathbb{R}^n$:

- The set $\{0\}$ containing just the zero vector.
- The set $\mathbb{R}^n$ itself.
- The set of all scalar multiples of a single vector.
- The span of any set of vectors in $\mathbb{R}^n$.
- The range of a linear function $T : \mathbb{R}^k \to \mathbb{R}^n$.
- The set of vectors v with $T(v) = 0$ for a linear function $T : \mathbb{R}^n \to \mathbb{R}^k$.

The union of two subspaces is not necessarily a subspace. (Consider two lines in $\mathbb{R}^2$)

The sum of two subspaces U and V is the set $U + V = \{u + v : u \in U \text{ and } v \in V\}$. This is a subspace.

The intersection of two subspaces is always a subspace. (Check the conditions defining a subspace.)

Definition. To any $m \times n$ matrix A there are two corresponding subspaces of interest:

1. The *column space* of A is the subspace $\operatorname{Col} A$ of $\mathbb{R}^m$ given by the span of the columns of A.
2. The *null space* of A is the subspace $\operatorname{Nul} A$ of $\mathbb{R}^n$ given by the set of vectors $v \in \mathbb{R}^n$ with $Av = 0$.

It is not obvious from these definitions, but it will turn out that each subspace of $\mathbb{R}^m$ occurs as the column space of some matrix. Likewise, each subspace of $\mathbb{R}^n$ occurs as the null space of some matrix.

If A and B are matrices with the same number of rows then $\operatorname{Col} \begin{bmatrix} A & B \end{bmatrix} = \operatorname{Col} A + \operatorname{Col} B$.

If A and B are matrices with the same number of columns then $\operatorname{Nul} \begin{bmatrix} A \\ B \end{bmatrix} = \operatorname{Nul} A \cap \operatorname{Nul} B$.

Definition. A *basis* of a subspace H of $\mathbb{R}^n$ is a set of linearly independent vectors whose span is H.

An important basis with its own notation: the *standard basis* of $\mathbb{R}^n$ consists of the vectors $e_1, e_2, \dots, e_n$ where e_i is the vector in $\mathbb{R}^n$ with 1 in row i and 0 in all other rows.

Theorem. Every subspace H of $\mathbb{R}^n$ has a basis of size at most n.

Let A be an $m \times n$ matrix.

How to find a basis of $\text{Nul}\, A$**.**

1. Find all solutions to $Ax = 0$ by row reducing A to echelon form. Recall that x_i is a *basic variable* if column i of $\mathsf{RREF}(A)$ contains a leading 1, and that otherwise x_i is a *free variable.*
2. Express each basic variable in terms of the free variables, and then write

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_{i_1} b_1 + x_{i_2} b_2 + \dots + x_{i_k} b_k$$

   where x_{i_1}, $x_{i_2}, \dots,$ x_{i_k} are the free variables and $b_1, b_2, \dots, b_k \in \mathbb{R}^n$.
3. The vectors $b_1, b_2, \dots, b_k$ then form a basis for $\text{Nul}\, A$.

Example. Suppose $A = \begin{bmatrix} 1 & 2 & 5 & 8 \\ 2 & 3 & 7 & 0 \end{bmatrix}$.

1. Then $A \sim \begin{bmatrix} 1 & 2 & 5 & 8 \\ 0 & -1 & -3 & -16 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -24 \\ 0 & 1 & 3 & 16 \end{bmatrix}$ so $Ax = 0$ if and only if

$$\begin{cases} x_1 - x_3 - 24x_4 = 0 \\ x_2 + 3x_3 + 16x_4 = 0 \end{cases} \quad \text{which means} \quad \begin{cases} x_1 = x_3 + 24x_4 \\ x_2 = -3x_3 - 16x_4. \end{cases}$$

2. Thus x_1, x_2 are basic variables, x_3, x_4 are free variables, and if $Ax = 0$ then

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 24x_4 \\ -3x_3 - 16x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 24 \\ -16 \\ 0 \\ 1 \end{bmatrix}.$$

3. The set of vectors $\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 24 \\ -16 \\ 0 \\ 1 \end{bmatrix} \right\}$ is then a basis for $\text{Nul}\, A$.

How to find a basis of $\text{Col}\, A$**.**

1. The pivot columns of A form a basis of $\text{Col}\, A$.

This looks simpler than the previous algorithm, but to find out which columns of A are pivot columns, we have to row reduce A to echelon form, which takes just as much work as finding a basis of $\text{Nul}\, A$.

Example. If $A = \begin{bmatrix} 1 & 2 & 5 & 8 \\ 2 & 3 & 7 & 0 \end{bmatrix}$ then columns 1, 2 have pivots so $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ is a basis for $\text{Col}\, A$.

This is not the only set of columns of A that forms a basis for $\text{Col}\, A$, however.

2 Coordinate systems

Suppose H is a subspace of $\mathbb{R}^n$. Let $b_1, b_2, \dots, b_k$ be a basis of H.

Theorem. Let $v \in H$. There are unique coefficients $c_1, c_2, \dots, c_k \in \mathbb{R}$ such that

$$c_1 b_1 + c_2 b_2 + \cdots + c_k b_k = v.$$

Proof. Since our basis spans H, there must be some coefficients with $c_1 b_1 + c_2 b_2 + \cdots + c_k b_k = v$. If these coefficients were not unique, so that we could write $c_1' b_1 + c_2' b_2 + \cdots + c_k' b_k = v$ for some different list of numbers $c_1', c_2', \dots, c_k' \in \mathbb{R}$, then we would have

$$\begin{aligned} 0 = v - v &= (c_1 b_1 + c_2 b_2 + \cdots + c_k b_k) - (c_1' b_1 + c_2' b_2 + \cdots + c_k' b_k) \\ &= (c_1 - c_1') b_1 + (c_2 - c_2') b_2 + \cdots + (c_k - c_k') b_k. \end{aligned}$$

In this case, since our numbers are different, at least one of the differences $c_i - c_i'$ must be nonzero, and so what we just wrote is a nontrivial linear dependence among the vectors $b_1, b_2, \dots, b_k$. But this is impossible since the elements of a basis are linearly independent. □

Let $\mathcal{B} = (b_1, b_2, \dots, b_k)$ be the list consisting of our basis vectors in some fixed order.

Given $v \in H$, define $[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$ as the unique vector with $c_1 b_1 + c_2 b_2 + \cdots + c_k v_k = v$.

Equivalently, $[v]_{\mathcal{B}}$ is the unique solution to the matrix equation $\begin{bmatrix} b_1 & b_2 & \cdots & b_k \end{bmatrix} x = v$.

We call $[v]_{\mathcal{B}}$ the *coordinate vector of v in the basis $\mathcal{B}$* or just *v in the basis $\mathcal{B}$*.

Example. If $H = \mathbb{R}^n$ and $\mathcal{B} = (e_1, e_2, \dots, e_n)$ is the standard basis then $[v]_{\mathcal{B}} = v$.

Example. If $H = \mathbb{R}^n$ and $\mathcal{B} = (e_n, \dots, e_2, e_1)$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then $[v]_{\mathcal{B}} = \begin{bmatrix} v_n \\ \vdots \\ v_2 \\ v_1 \end{bmatrix}$.

Example. Let $b_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$.

Then $\mathcal{B} = (b_1, b_2)$ is a basis for $H = \mathbb{R}\text{-span}\{b_1, b_2\}$, which is a subspace of $\mathbb{R}^3$.

The unique $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ such that $\begin{bmatrix} 3 & -1 \\ 6 & 0 \\ 2 & 1 \end{bmatrix} x = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ is found by row reduction:

$$\begin{bmatrix} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 1 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}.$$

The last matrix implies that $x_1 = 2$ and $x_2 = 3$ so $[v]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

Example. If $b_1 = e_1 - e_2$, $b_2 = e_2 - e_3$, $b_3 = e_3 - e_4$, ..., $b_{n-1} = e_{n-1} - e_n$ and

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ -v_1 - v_2 - \cdots - v_{n-1} \end{bmatrix}$$

then $v \in H = \mathbb{R}\text{-span}\{b_1, b_2, \dots, b_{n-1}\}$ and

$$[v]_\mathcal{B} = \begin{bmatrix} v_1 \\ v_1 + v_2 \\ v_1 + v_2 + v_3 \\ v_1 + v_2 + v_3 + v_4 \\ \vdots \\ v_1 + v_2 + v_3 + \cdots + v_{n-1} \end{bmatrix} \in \mathbb{R}^{n-1}.$$

The notation $[v]_\mathcal{B}$ gives us an easy way to check the following important property:

Theorem. Let H be a subspace of $\mathbb{R}^n$. Then all bases of H have the same number of elements.

Proof. Suppose $\mathcal{B} = (b_1, b_2, \dots, b_k)$ and $\mathcal{B}' = (b'_1, b'_2, \dots, b'_l)$ are two (ordered) bases of H with $k < l$.

Then $[b'_1]_\mathcal{B}$, $[b'_2]_\mathcal{B}$, ..., $[b'_l]_\mathcal{B}$ are $l > k$ vectors in $\mathbb{R}^k$, so they must be linearly dependent.

This means there exist coefficients $c_1, c_2, \dots, c_l \in \mathbb{R}$, not all zero, with

$$c_1[b'_1]_\mathcal{B} + c_2[b'_2]_\mathcal{B} + \cdots + c_l[b'_l]_\mathcal{B} = 0.$$

But we have $c_1[b'_1]_\mathcal{B} + c_2[b'_2]_\mathcal{B} + \cdots + c_l[b'_l]_\mathcal{B} = [c_1b'_1 + c_2b'_2 + \cdots + c_lb'_l]_\mathcal{B}$.

(This is the key step; why is this true? Think about how $[v]_\mathcal{B}$ is defined.)

Thus $[c_1b'_1 + c_2b'_2 + \cdots + c_lb'_l]_\mathcal{B} = 0$, so

$$c_1b'_1 + c_2b'_2 + \cdots + c_lb'_l = \begin{bmatrix} b_1 & b_2 & \cdots & b_k \end{bmatrix} [c_1b'_1 + c_2b'_2 + \cdots + c_lb'_l]_\mathcal{B} = 0.$$

(The first equality holds since by definition $v = \begin{bmatrix} b_1 & b_2 & \cdots & b_k \end{bmatrix} [v]_\mathcal{B}$.)

Since the coefficients c_i are not all zero, this contradicts the fact that $b'_1, b'_2, \dots, b'_l$ are linearly independent. This means our original assumption that H has two bases of different sizes is impossible. □

3 Dimension

Let $\mathcal{B} = (b_1, b_2, \dots, b_k)$ be an ordered basis of a subspace H of $\mathbb{R}^n$.

The function $H \to \mathbb{R}^k$ with the formula $v \mapsto [v]_\mathcal{B}$ is linear and invertible.

Thus H "looks the same as" $\mathbb{R}^k$.

For this reason we say that H is *k-dimensional*. More generally:

Definition. The *dimension* of a subspace H is the number of vectors in any basis of H.

We denote the dimension of H by $\dim H$. This number belongs to $\{0, 1, 2, 3, \dots\}$.

The only way we can have $\dim H = 0$ is if $H = \{0\}$ is the zero subspace.

Example. We have $\dim \mathbb{R}^n = n$.

If H is the set of all vectors of the form $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$, then H is a subspace and $\dim H = k$.

Note that $e_1, e_2, \dots, e_k$ is a basis for H.

A line in $\mathbb{R}^2$ through the origin is a 1-dimensional subspace.

Let A be an $m \times n$ matrix.

The processes we gave to construct bases of $\operatorname{Nul} A$ and $\operatorname{Col} A$ imply that:

Corollary. The dimension of $\operatorname{Nul} A$ is the number of free variables in the linear system $Ax = 0$.

Corollary. The dimension of $\operatorname{Col} A$ is the number of pivot columns in A.

There is a special name for the dimension of the column space of a matrix:

Definition. The *rank* of a matrix A is $\operatorname{rank} A = \dim \operatorname{Col} A$.

Putting everything together gives the following pair of important results.

Theorem (Rank-nullity theorem)**.** If A is a matrix with n columns then $\operatorname{rank} A + \dim \operatorname{Nul} A = n$.

Proof. The number of free variables in the system $Ax = 0$ is also the number non-pivot columns in A.

Therefore $\operatorname{rank} A + \dim \operatorname{Nul} A$ is the total number of columns in A. □

Theorem (Basis theorem)**.** If H is a subspace of $\mathbb{R}^n$ with $\dim H = p$ then

1. Any set of p linearly independent vectors in H is a basis for H.
2. Any set of p vectors in H whose span is H is a basis for H.

Proof. Suppose we have p linearly independent vectors in H. If these vectors do not span H, then adding a vector which is in H but not in their span gives a set of $p + 1$ linearly independent vectors in H.

If this larger set still does not span H, then adding a vector from H that is not in the span gives an even larger linearly independent set of $p + 2$ vectors.

Continuing in this way must eventually produce a basis for H, but this basis will have more than p elements, contradicting $\dim H = p$.

Suppose we instead have p vectors whose span is H. If these vectors are linearly dependent, then one of the vectors is a linear combination of the others. Remove this vector to get $p - 1$ vectors that span H.

If these vectors are still not linearly independent, then one is a linear combination of the others and removing this vector gives a set of $p - 2$ vectors that span H.

Continuing in this way must eventually produce a basis for H, but this basis will have fewer than p elements, contradicting $\dim H = p$. □

Corollary. If H is an n-dimensional subspace of $\mathbb{R}^n$ then $H = \mathbb{R}^n$.

Proof. If H has a basis with n elements then these elements are linearly independent, so form a basis for $\mathbb{R}^n$. Then every vector in $\mathbb{R}^n$ is a linear combination of the basis vectors, so belongs to H. □

If U and V are two sets then we write "$U \subset V$" or "$U \subseteq V$" to mean that every element of U is also an element of V. Both notations mean the same thing. If $U \subseteq V$ then it could be true that $U = V$.

On the other hand, writing "$U \subsetneq V$" means "$U \subseteq V$ but $U \neq V$."

It holds that $U = V$ if and only if we have both $U \subseteq V$ and $V \subseteq U$.

Corollary. If $U, V \subseteq \mathbb{R}^n$ are subspaces with $U \subseteq V$ but $U \neq V$, then $\dim U < \dim V \leq n$.

Proof. If $j = \dim V \leq \dim U = k$ and $u_1, u_2, \dots, u_k$ is a basis for U, then $u_1, u_2, \dots, u_j$ would be linearly independent and therefore a basis for V. But then $V \subseteq U$ which would imply $U = V$ if also $U \subseteq V$. □

Corollary. Let A be an $n \times n$ matrix. The following are equivalent:

(a) A is invertible.

(b) The columns of A form a basis for $\mathbb{R}^n$.

(c) $\operatorname{rank} A = \dim \operatorname{Col} A = n$.

(d) $\dim \operatorname{Nul} A = 0$.

Proof. We have already seen that (a) and (b) are equivalent.

(c) holds if and only if the columns of A span $\mathbb{R}^n$ which is equivalent to (a).

(d) holds if and only if the columns of A are linearly independent which is equivalent to (a). □

4 Vocabulary

Keywords from today's lecture:

1. **Coordinate vector** of a vector $v \in H$ with respect to an ordered basis $\mathcal{B} = (b_1, b_2, \dots, b_k)$.

   The unique vector of coefficients $[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix} \in \mathbb{R}^k$ with $c_1 b_1 + c_2 b_2 + \cdots + c_k b_k = v$.

   Example: If $H = \mathbb{R}^2$ and $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$ and $v = \begin{bmatrix} x \\ y \end{bmatrix}$ then $[v]_{\mathcal{B}} = \begin{bmatrix} x - y \\ y \end{bmatrix}$.

2. **Dimension** of a subspace $H \subseteq \mathbb{R}^n$

   The number $\dim H$ of vectors in any basis for H.

3. **Rank** of an $m \times n$ matrix A.

   The dimension of the column space $\operatorname{Col} A$. This is also the number of pivot columns in A.

   This is denoted $\operatorname{rank} A$.

4. **Rank-nullity theorem**.

   If A is an $m \times n$ matrix then $\dim \operatorname{Col} A + \dim \operatorname{Nul} A = \operatorname{rank} A + \dim \operatorname{Nul} A = n$.

5. **Basis theorem**.

   If $H \subseteq \mathbb{R}^n$ is a subspace with $\dim H = p$ then (1) any set of p linearly independent vectors in H is a basis for H and (2) any set of p vectors whose span is H is a basis for H.