Instructions: Choose **2 problems** and write down detailed solutions, showing all necessary work. You can earn up to **8 extra credit points** by correctly solving additional problems.[1]

Feel free to discuss problems with other students but write up your own solutions. If your solutions appear to be copied from somewhere else, you will automatically receive zero credit.

To get full credit for the offline homework, you just need to make a good-faith attempt on two problems. The bar for receiving extra credit points is higher: your solutions need to be close to completely correct.

1. Warmup: find general formulas for the solutions of the linear systems

$$\begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + 4x_2 = 3 \end{cases} \qquad \text{and} \qquad \begin{cases} x_1 + 2x_2 = 1 \\ 3x_1 + 6x_2 = 3. \end{cases}$$

   Now suppose a, b, c, d, p, q are real numbers with $ad - bc \neq 0$.

   Find a general formula for the solution of the linear system $\begin{cases} ax_1 + bx_2 = p \\ cx_1 + dx_2 = q. \end{cases}$

2. Suppose n is a positive integer and $a_1, a_2, \dots, a_n$ are real numbers.

   What condition must be satisfied for the linear system

$$\begin{cases} x_1 - x_2 = a_1 \\ x_2 - x_3 = a_2 \\ x_3 - x_4 = a_3 \\ \vdots \\ x_{n-1} - x_n = a_{n-1} \\ x_n - x_1 = a_n \end{cases}$$

   to be consistent? Write down the augmented matrix of this linear system. Assuming the system is consistent, find a general formula for its solutions.

3. Suppose we have two linear systems with the same number of equations and the same number of variables. Then the systems' augmented matrices have the same size. If the augmented matrices are row equivalent then the systems are equivalent, meaning they have the same solutions. Do there exist two linear systems, both with m equations and n variables, that are equivalent but whose augmented matrices are **not** row equivalent? Explain why this is impossible or find an example.

4. A *quadratic equation* in two variables x_1, x_2 is an equation of the form

$$ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 = f$$

   for real numbers a, b, c, d, e, f. A *quadratic system* is a list of quadratic equations.

   While the set of solutions (x_1, x_2) to a linear equation (in two variables) forms a line in the Cartesian plane, the set of solutions to a quadratic equation (in two variables) forms a **conic section** in the Cartesian plane. A conic section is a curve given by an **ellipse**, **parabola**, or **hyperbola**. A circle is a special case of an ellipse and a straight line is considered to be a degenerate case of a parabolic. See `https://en.wikipedia.org/wiki/Conic_section` for more information about what these shapes look like.

   What are the possibilities for the number of solutions to a quadratic system in two variables? (In this problem, a **solution** means a real-valued solution (x_1, x_2) with $x_1, x_2 \in \mathbb{R}$.) Justify your answer by adapting the geometric argument in Lecture 1 that was used to prove that every linear system in

[1] There will be ∼11 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼88 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

two variables has 0, 1, or infinitely many solutions. Draw a picture corresponding to each different possibility for the number of solutions.

5. There is a way to multiply two 2×2 matrices to get another 2×2 matrix. The formula is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{bmatrix}.$$

   (a) Find a 2×2 matrix I with $\begin{bmatrix} a & b \\ c & d \end{bmatrix} I = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for all $a, b, c, d \in \mathbb{R}$. What is $I \begin{bmatrix} a & b \\ c & d \end{bmatrix}$?

   (b) Find a 2×2 matrix E with $E \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} y & z \\ w & x \end{bmatrix}$ for all $w, x, y, z \in \mathbb{R}$. What is $\begin{bmatrix} w & x \\ y & z \end{bmatrix} E$?

   (c) Find a 2×2 matrix J with $J^2 = -I$ where we define $-\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$.

6. If A is a 1×3 matrix then $\mathsf{RREF}(A)$ either has the form

$$\begin{bmatrix} 1 & * & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & * \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

   where each $*$ means an arbitrary real number. In the first, second, and fourth cases, A is the augmented matrix of a linear system in two variables with infinitely many solutions. In the third case, A is the augmented matrix of a linear system in two variables with zero solutions.

   Describe with similar notation what the possibilities are for $\mathsf{RREF}(A)$ if A is a 2×3 matrix. In each case, indicate how many solutions there are for the linear system whose augmented matrix is A.

7. What are the possibilities for $\mathsf{RREF}\left(\begin{bmatrix} x & 1 \\ y & 2 \end{bmatrix}\right)$ if x and y are arbitrary real numbers? Draw a picture of the xy-plane in which you identify the regions of points (x, y) where $\mathsf{RREF}\left(\begin{bmatrix} x & 1 \\ y & 2 \end{bmatrix}\right)$ takes its different possible values.

8. If the reduced echelon form of the augmented matrix of some linear system is

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 5 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

   then what is the general formula for the solution to the linear system?

   Going in the opposite direction, find a matrix A that is the augmented matrix of a linear system with 3 equations and 4 variables whose general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1-3a-4b \\ a \\ 2-5b \\ b \end{bmatrix} \text{ for all } a, b \in \mathbb{R}.$$

9. Randomly choose 9 distinct integers and use these as the entries of a 3×3 matrix A. Compute $\mathsf{RREF}(A)$ by hand without using a calculator, showing all of your work in the intermediate steps of the row reduction algorithm. (After you've done this, feel free to use a calculator to check your answer.) What is the smallest number of entries in A that you can change to form a 3×3 matrix B with $\mathsf{RREF}(A) \neq \mathsf{RREF}(B)$?

10. "Almost all linear systems with n equations and n variables have unique solutions." (a) Why is this statement plausible when $n = 1$? (b) How would you justify this statement when $n = 2$?