

**Instructions:** Choose **2 problems** and write down detailed solutions, showing all necessary work. You can earn up to **8 extra credit points** by correctly solving additional problems.<sup>1</sup>

Feel free to discuss problems with other students but write up your own solutions. If your solutions appear to be copied from somewhere else, you will automatically receive zero credit.

To get full credit for the offline homework, you just need to make a good-faith attempt on two problems. The bar for receiving extra credit points is higher: your solutions need to be close to completely correct.

1. Assume  $h$  and  $k$  are real numbers and consider the linear system

$$\begin{aligned}x_1 + 3x_2 &= k \\4x_1 + hx_2 &= 7.\end{aligned}$$

Determine all values of  $h$  and  $k$  such that this system has (i) zero solutions, (ii) a unique solution, or (iii) infinitely many solutions.

2. Suppose  $v_1 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$ ,  $v_2 = \begin{bmatrix} -3 \\ 2 \\ 7 \end{bmatrix}$ ,  $v_3 = \begin{bmatrix} 3 \\ -6 \\ -9 \end{bmatrix}$ , and  $w = \begin{bmatrix} 6 \\ -10 \\ a \end{bmatrix}$  where  $a \in \mathbb{R}$ .

Find all values of  $a$  such that  $w$  is in  $\mathbb{R}\text{-span}\{v_1, v_2, v_3\}$ .

3. Describe all matrices  $A$  such that  $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$  and  $A \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 1 \end{bmatrix}$ .

4. Assume  $A$  is a  $3 \times 3$  matrix. Only the first and second columns of  $A$  are pivot columns and

$$A \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

What is  $\text{RREF}(A)$ ? What is the general solution to  $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ?

5. Find all values of  $h \in \mathbb{R}$  for which the vectors

$$\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} -5 \\ 5 \\ 11 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ h \end{bmatrix}$$

are linearly dependent.

6. Suppose  $v_1, v_2, \dots, v_k \in \mathbb{R}^n$  are linearly independent vectors.

- (a) If we add another vector  $v_{k+1}$  to this list, will it always still be linearly independent?

What can you say about when the larger list of vectors is still linearly independent?

- (b) If we delete one of the vectors from the list, say  $v_k$ , will it always still be linearly independent?

What can you say about when the smaller list of vectors is still linearly independent?

Explain and justify your answers to both parts.

7. Choose  $m$  vectors  $v_1, v_2, \dots, v_m \in \mathbb{R}^n$ .

Suppose  $\mathbb{R}\text{-span}\{v_1, v_2, \dots, v_m\} = \mathbb{R}^n$ . A theorem in Lecture 4 states that this occurs if and only if the matrix  $A = \begin{bmatrix} v_1 & v_2 & \dots & v_m \end{bmatrix}$  has a pivot position in every row.

Use this fact to explain why we must have  $m \geq n$ .

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<sup>1</sup> There will be  $\sim 11$  weeks of assignments, each with  $\sim 10$  practice problems, so you can earn up to  $\sim 88$  equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

$$8. \text{ Let } V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n : a_1 + a_2 + \cdots + a_n = 0 \right\}.$$

Find a set of  $n - 1$  vectors in  $\mathbb{R}^n$  with span equal to  $V$ .

Then use the previous exercise to show that no set of  $n - 2$  vectors in  $\mathbb{R}^n$  has span equal to  $V$ .

*Hint: show if we have a set spanning  $V$  then we can add 1 vector to get a set spanning  $\mathbb{R}^n$ .*

9. Choose an angle  $\theta \in [0, 2\pi)$ .

What is the relationship between the vector  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2$  and  $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ ?

Draw a picture and justify your answer.

Then use your answer to prove the trigonometric sum-angle formulas

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) \quad \text{and} \quad \sin(\theta + \phi) = \cos(\theta)\sin(\phi) + \sin(\theta)\cos(\phi).$$

*Hint: answer the question for  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$  and  $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ v_2 \end{bmatrix}$ , using the unit circle definitions of  $\sin$  and  $\cos$ . Then use parallelogram rule for vector addition to deduce the general answer.*

10. Suppose  $a$  and  $b$  are real numbers. Consider the lines

$$\mathcal{L}_1 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : x_2 = ax_1 \right\} \quad \text{and} \quad \mathcal{L}_2 = \left\{ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \mathbb{R}^2 : y_2 = by_1 \right\}.$$

Find all values of  $a$  and  $b$  such that there is exactly one way of writing

$$\begin{bmatrix} 2121 \\ 2022 \end{bmatrix} = v + w$$

with  $v \in \mathcal{L}_1$  and  $w \in \mathcal{L}_2$ . Find a formula for  $v$  and  $w$  in this case.