Instructions: *Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.*

Due on **Thursday, April 21**.

Let Λ *and* Λ^+ *be the sets of* weights *and* fundamental weights *for a root system* Φ *with Weyl group* W *as defined in* §13 *of the textbook. Consult that section of the textbook to do the following exercises.*

1. Suppose $\lambda \in \Lambda^+$ and $\sigma \in W$. Let $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ be half the sum of the positive roots in Φ relative to an arbitrary simple system. Prove that $\sigma(\lambda+\delta)-\delta \in \Lambda^+$ if and only if $\sigma = 1$.
2. Prove that each subset of Λ is contained in a unique smallest saturated set (in the sense defined in §13.4), which is finite if the subset in question is finite.

Below, let L *be a Lie algebra and write* $\mathfrak{U}(L)$ *for its universal enveloping algebra.*

3. Prove that if $\dim L < \infty$ then $\mathfrak{U}(L)$ has no zero divisors.
4. If $X \in L$ then extend $\operatorname{ad} X$ to an endomorphism of $\mathfrak{U}(L)$ by setting $\operatorname{ad} X(Y) = XY - YX$ for $Y \in \mathfrak{U}(L)$. Prove that if $\dim L < \infty$ then each element of $\mathfrak{U}(L)$ belongs to a finite-dimensional L-submodule with respect to this adjoint action.
5. If L is a free Lie algebra on a set X, prove that $\mathfrak{U}(L)$ is isomorphic to the tensor algebra on a vector space having X as a basis.
6. Describe the free Lie algebra on a set $X = \{x\}$ of size one.