MATH 5143 - Lecture #11

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Clarifications about universal enveloping algebras

L Lie algebra

$T(L)$ tensor algebra (assoc.) alg. but noncommutative

$S(L)$ symmetric algebra (assoc.) alg. that is commutative

$\mathcal{U}(L)$ univ. enveloping algebra (assoc.) alg. also NOT commutative

What's true: the associated graded algebra of $\mathcal{U}(L)$ is $\cong S(L)$

in $\mathcal{U}(L)$ we have $\underbrace{XY}_{\text{deg } 2} = \underbrace{YX}_{\text{deg } 2} + \underbrace{[X,Y]}_{\text{deg } 1} \neq YX$

key point: multiplication of homogeneous tensors in $\mathcal{U}(L)$ IGNORING LOWER ORDER TERMS is commutative

Last time $\mathbb{F}$ is an alg. closed field of characteristic zero

Fix a root system Φ with simple roots $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$

Define L to be the Lie algebra / $\mathbb{F}$ generated by

$$x_1, x_2, \dots, x_n, \quad y_1, y_2, \dots, y_n, \quad h_1, h_2, \dots, h_n$$

subject to relations

(S1) $[h_i, h_j] = 0$

(S2) $[x_i, y_j] = \delta_{ij} h_i$

(S3) $[h_i, x_j] = \langle \alpha_j, \alpha_i \rangle x_j$

$[h_i, y_j] = -\langle \alpha_j, \alpha_i \rangle y_j$

$(S_{ij}^{+,-})$ if $i \neq j$

$(\operatorname{ad} x_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(x_j) = 0$

$(\operatorname{ad} y_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(y_j) = 0$

Thm (Serre) L is a finite-dim semisimple Lie algebra with Cartan subalgebra $H \stackrel{\text{def}}{=} \mathbb{F}\text{-span}\{h_1, h_2, \dots, h_n\}$ and root system $\cong \Phi$. Also any semisimple Lie algebra with root system $\cong \Phi$ is (naturally) $\cong L$.

Prop If L is a nonzero Lie algebra of $\mathfrak{gl}(V)$ (resp., $\mathfrak{sl}(V)$) where $\dim V < \infty$ and V is an irreducible L-module then L is reductive (resp. semisimple)

[otherwise not simple]

$n \geq 2$ $\quad$ $n \geq 3$ $\quad$ $n \neq 4$

Cor Each classical Lie algebra $\mathfrak{sl}_n(F)$, $\mathfrak{sp}_{2n}(F)$, or $\mathfrak{o}_n(F)$ is a semisimple (and simple) subalgebra of $\mathfrak{sl}(V)$ (for $V = F^n$ or F^{2n}).

because the root systems have connected Dynkin diagrams

Representation theory setup: L is a semisimple fin. dim. Lie algebra / $\mathbb{F}$

$H \leq L$ is a Cartan subalgebra, $\Phi \subset H^*$ is the root system,

$\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_n\} \subset \Phi$ is a chosen base, $\Phi^+ = \{\text{positive roots}\}$,

and $W = \langle r_\alpha \mid \alpha \in \Phi \rangle = \langle r_\alpha \mid \alpha \in \Delta \rangle \leq GL(H^*)$

First key observation: Any finite-dim L-module V decomposes as a direct sum of weight spaces $V = \bigoplus_{\lambda \in H^*} V_\lambda$ where $V_\lambda \overset{\text{def}}{=} \{v \in V \mid h \cdot v = \lambda(h) v \ \forall h \in H\}$

Call $\lambda \in H^*$ a weight of V if $V_\lambda \neq 0$ and call V_λ a weight space.

We make same definitions if $\dim V = \infty$ but in that case the sum of weight spaces $\bigoplus_{\lambda \in H^*} V_\lambda$ [which is always direct] may be $\subsetneq V$.

Idea: to avoid pathologies with arbitrary infinite-dim L-modules, we consider standard cyclic modules.

A maximal vector of weight $\lambda \in H^*$ in an arbitrary L-module V is a vector $0 \neq v^+ \in V$ with $\boxed{x v^+ = 0 \quad \forall x \in L_\alpha \quad \forall \alpha \in \Delta}$

Lie's theorem (applied to the Borel algebra $B = H \oplus \bigoplus_{\alpha \in \Phi^+} L_\alpha$ acting on V) ensures that V has a maximal vector whenever $\dim V < \infty$.

Def A standard cyclic module for L is an L-module V with a maximal vector v^+ such that $V = \mathcal{U}(L) \cdot v^+$.

In this situation, the maximal vector v^+ will belong to V_λ for some $\lambda \in H^*$, and we say V has highest weight λ and we call v^+ is a highest weight vector

Thm (Structure thm for standard cyclic modules)

For each $\beta \in \Phi^+$ choose $x_\beta \in L_\beta$, $y_\beta \in L_{-\beta}$ such that $[x_\beta, y_\beta] = h_\beta \in H$

Write $\lambda > \mu$ for $\lambda, \mu \in H^*$ if $\lambda - \mu \in \mathbb{Z}_{\geq 0}\text{-span}\{\alpha \in \Delta\}$

Suppose V is standard cyclic L-module with maximal vector $v^+ \in V_\lambda$

(a) If $\Phi^+ = \{\beta_1, \beta_2, \beta_3, \dots\}$ then V is spanned by vectors of the form

$$y_{\beta_{i_1}} y_{\beta_{i_2}} y_{\beta_{i_3}} \cdots y_{\beta_{i_k}} \quad \text{where} \quad 1 \leq i_1 \leq i_2 \leq i_3 \leq \dots \leq i_k$$

(b) All weights μ for V have $\mu < \lambda$ and $\dim V_\mu < \infty$ and $\dim V_\lambda = 1$.

(c) Any submodule of V is a direct sum of its weight spaces

(d) V is indecomposable with unique maximal proper submodule and unique irreducible quotient

(e) Every homomorphic image of V is standard cyclic of same weight λ

(f) If V is irreducible then all maximal vectors are nonzero scalar multiples of $v^+ \in V$.

Today: first, two more thms about standard cyclic modules

Thm A If V and W are irreducible standard cyclic L-modules with same highest weight $\lambda \in H^*$ then $V \cong W$

Thm B If $\lambda \in H^*$ then there exists an irreducible standard cyclic L-module $V(\lambda)$ of highest weight λ.

Pf of Thm A. Let $X = V \oplus W \overset{\text{def}}{=} \{v + w \mid v \in V, w \in W\}$. This is an L-module and if $v^+ \in V$ and $w^+ \in W$ are highest weight vectors then $x^+ \overset{\text{def}}{=} v^+ + w^+ \in X$ is a maximal vector also of weight λ.

Let Y be the submodule of X generated by x^+. This is standard cyclic by def. But $V \cong Y/\ker \pi_1$ and $W \cong Y/\ker \pi_2$ where $\pi_1 : Y \to V$ and $\pi_2 : Y \to W$ are the obvious surjective homomorphisms. This means V and W are both isomorphic to the unique irreducible quotient of Y. $\square$

To prove Thm B we must explain how to construct standard cyclic modules

<u>Induced modules</u> Begin with a 1-dim vector space $D_\lambda = \mathbb{F}\text{span}\{v^+\}$ spanned by some vector v^+. Let $\lambda \in H^*$ and $B = B(\Delta) \stackrel{\text{def}}{=} H \oplus \bigoplus_{\alpha \in \Phi^+} L_\alpha \subset L$.

The <u>Borel subalgebra</u> B acts on D_λ linearly by

$$h \cdot v^+ \stackrel{\text{def}}{=} \lambda(h) v^+ \quad \text{and} \quad x v^+ \stackrel{\text{def}}{=} 0 \quad \text{for } h \in H,\ \alpha \in \Phi^+,\ x \in L_\alpha$$

This makes D_λ into a module for B and for $\mathcal{U}(B)$

<u>Def</u> Let $Z(\lambda) \stackrel{\text{def}}{=} \mathcal{U}(L) \otimes_{\mathcal{U}(B)} D_\lambda$

This a general construction of a $\mathcal{U}(L)$-module: $\mathcal{U}(L)$ + D_λ

$\mathcal{U}(L)$: left $\mathcal{U}(L)$-mod, right $\mathcal{U}(B)$-mod; D_λ: left $\mathcal{U}(B)$-mod

Concretely, $Z(\lambda)$ is vector space spanned by the tensors $x \otimes y$ ($x \in \mathcal{U}(L)$, $y \in D_\lambda$) subject to relations

$$(x+x') \otimes y = x \otimes y + x' \otimes y \qquad c(x \otimes y) = (cx) \otimes y = x \otimes (cy)$$

$$x \otimes (y+y') = x \otimes y + x \otimes y' \qquad xb \otimes y = x \otimes by \quad \text{for } x \in \mathcal{U}(L),\ b \in \mathcal{U}(B),\ y \in D_\lambda$$

The way L acts on $Z(\lambda)$ is $A\cdot(x\otimes y) \stackrel{\text{def}}{=} (Ax)\otimes y$ (w/ highest weight vector $1\otimes v^+$)

Claim $Z(\lambda)$ is a standard cyclic L-module of weight λ.

Pf Every $y\in D_\lambda$ is a scalar multiple of v^+ so every tensor $x\otimes y\in Z(\lambda)$ is equal to $\tilde{x}\cdot(1\otimes v^+)$ where $\tilde{x}\in\mathfrak{U}(L)$ is a scalar multiple of x. For $x\in L_\alpha$ for $\alpha\in\Delta$ we have $x\in B$ so

$x\cdot(1\otimes v^+) = x\otimes v^+ = 1\otimes xv^+ = 1\otimes 0 = 0$. Also for $h\in H\subset B$

we have $h\cdot(1\otimes v^+) = h\otimes v^+ = 1\otimes hv^+ = 1\otimes\lambda(h)v^+ = \lambda(h)(1\otimes v^+)$ □

Let $N^- = \bigoplus_{\alpha \in -\Phi^+} L_\alpha$. The relation $xb \otimes v^+ = x \otimes bv^+$ $\forall b \in B$, since $L = N^- \oplus B$,

implies that if $\Phi^+ = \{\beta_1, \beta_2, \beta_3, \dots\}$ and $\left\{ \begin{array}{l} x_i = x_{\beta_i} \text{ spans } L_{\beta_i} \\ y_i = y_{\beta_i} \text{ spans } L_{-\beta_i} \end{array} \right\}$ then

$$\{ y_{i_1} y_{i_2} \cdots y_{i_k} \otimes v^+ \mid k \geq 0 \text{ and } i_1 \leq i_2 \leq \cdots \leq i_k \}$$

is a basis for $Z(\lambda)$, via the PBW theorem.

Prop $Z(\lambda) \cong \mathcal{U}(L) / I(\lambda)$ as $\mathcal{U}(L)$-modules, where $I(\lambda)$ is left ideal generated in $\mathcal{U}(L)$ by the elements

$$\{x_1, x_2, x_3 \dots\} \cup \{ h_\alpha - \lambda(h_\alpha) \cdot 1 \mid \alpha \in \Phi \}$$

Pf These generators annihilate $1 \otimes v^+$ so there is a surjective morphism $\mathcal{U}(L)/I(\lambda) \to Z(\lambda)$ which is injective using PBW theorem. $\square$

Thm (Thm B) Define $V(\lambda)$ for $\lambda \in H^*$ to be the unique irreducible quotient of the standard cyclic module $Z(\lambda)$.

Then $V(\lambda)$ is standard cyclic of weight λ and irreducible.

Note: $V(\lambda)$ still might be infinite-dimensional

Pf Since $Z(\lambda)$ is standard cyclic, and since $V(\lambda)$ as a quotient is a homomorphic image of $Z(\lambda)$, everything follows from structure theorem for standard cyclic modules. $\square$

In some sense, hardest part of thm is showing $Z(\lambda) \neq 0$ (but we will not discuss this issue in detail, follows from PBW thm)

Two new goals: ① Explain when $V(\lambda)$ is finite dim.

② Determine weight spaces $V(\lambda)_\mu \subseteq V(\lambda)$

<u>Fact</u> If V is any irreducible L-module with $\dim V < \infty$ then $V \cong V(\lambda)$ for some $\lambda \in H^*$.

<u>Pf</u> If $\dim V < \infty$ then Lie's thm applied to B-action on V implies existence of a maximal vector of some weight λ. This vector must generate V by irreducibility, so $V \cong V(\lambda)$ by Thm A. □

For each simple root $\alpha_i \in \Delta$ let $S_i = S_{\alpha_i} = L_{-\alpha_i} \oplus \mathbb{F} h_{\alpha_i} \oplus L_{\alpha_i} \cong \mathfrak{sl}_2(\mathbb{F})$

Then $V(\lambda)$ is a module for S_i and a maximal vector for L is also maximal for S_i

<u>Thm</u> If $V \cong V(\lambda)$ and $\dim V < \infty$ then $\lambda(h_{\alpha_i}) \in \mathbb{Z}_{\geq 0}$ $\forall \alpha_i \in \Delta$ and if $\mu \in H^*$ is any weight for V then $\mu(h_{\alpha_i}) \in \mathbb{Z}$ $\forall \alpha_i \in \Delta$

<u>Pf sketch</u> Follows from $\mathfrak{sl}_2$-repn theory, as V decomposes as sum of fin. dim irr. S_i-modules.

Call $\lambda \in H^*$ $\begin{cases} \underline{\text{dominant}} \text{ if } \lambda(h_\alpha) \geq 0 \ \forall \alpha \in \Delta \ (\text{equiv. } \forall \alpha \in \Phi^+) \\ \underline{\text{integral}} \text{ if } \lambda(h_\alpha) \in \mathbb{Z} \ \forall \alpha \in \Delta \ (\text{equiv. } \forall \alpha \in \Phi) \end{cases}$

Then $\lambda \in H^*$ is <u>dominant integral</u> if $\lambda(h_\alpha) \in \mathbb{Z}_{\geq 0} \ \forall \alpha \in \Delta$

Let Λ be abelian group of integral weights and Λ^+ the subset of dominant integral weights. Note that $\Lambda \supset \Phi$.

For an L-module V let $\Pi(V) \subseteq H^*$ be its set of weights and define $\Pi(\lambda) = \Pi(V(\lambda))$. If $\dim V < \infty$ then $\Pi(\lambda) \subset \Lambda$.

<u>Next main thm</u> Suppose $\lambda \in \Lambda^+$. Then $V(\lambda)$ has finite dimension and the Weyl group $W \subseteq GL(H^*)$ permutes $\Pi(\lambda)$ with $\dim V(\lambda)_\mu = \dim V(\lambda)_{\sigma\mu}$ $\forall \sigma \in W$.

<u>Cor</u> The map $\lambda \mapsto V(\lambda)$ is a bijection from Λ^+ to isomorphism classes of irreducible fin. dim. L-modules.

<u>Pf</u> Combine main thm with fact and thm on prev slide (along with Thm A) □

Pf sketch of main thm

Some identities in $\mathcal{U}(L)$: writing $x_i = x_{\alpha_i}$, $y_i = y_{\alpha_i}$, $h_i = h_{\alpha_i}$ for $\alpha_i \in \Delta$

(a) $[x_j, y_i^{k+1}] = 0$ when $i \neq j$, $k \geq 0$

(b) $[h_j, y_i^{k+1}] = -(k+1)\,\alpha_i(h_j)\, y_i^{k+1}$ $\quad (k \geq 0)$

(c) $[x_i, y_i^{k+1}] = -(k+1)\, y_i^k (k - h_i)$ $\quad (k \geq 0)$

Straightforward algebra by induction on $k \geq 0$.

Now we derive a series of claims.

Claim (1) $y_i^{m_i+1} v^+ = 0$ where $m_i = \lambda(h_i) \in \mathbb{Z}_{\geq 0}$, and $v^+ \in V = V(\lambda)$ is a highest weight vector.

Pf Otherwise can use (a)-(c) to show that $y_i^{m_i+1} v^+$ is a second maximal vector of weight $\neq \lambda$ which is impossible $\square$

Claim (2) V contains a nonzero fin. dim. $S_i = S_{\alpha_i}$-module $\cong sl_2(F)$

Pf Consider subspace spanned by $v^+, y_i v^+, y_i^2 v^+, \ldots$

This is finite-dim by claim (1). $\square$

Claim (3) V is a sum of finite-dim S_i-modules

Pf Let V' be the sum of all S_i-submodules of finite dim in V

Then $V' \neq 0$ by claim (2). Check that V' is an L-module, hence $V' = V$

since V irreducible. $\square$

Claim (4) If $\phi : L \to gl(V)$ is repn corresp. to L-module structure on V

then $\phi(x_i)$ and $\phi(y_i)$ are both locally nilpotent (meaning nilpotent

when restricted to a finite-dim subspace)

Pf Each $v \in V$ is in a finite sum of fin. dim. S_i-modules, on which $\phi(x_i), \phi(y_i)$ act as

nilpotent operators, by sl_2-repn theory. $\square$

Claim (5) Define $\sigma_i \stackrel{\text{def}}{=} \exp(x_i)\exp(-y_i)\exp(x_i)$.

This is an automorphism of V (as a vector space)

Pf Just need to check that σ_i is well-defined, but this follows from prev claim. $\square$

Claim (6) If μ is a weight of V then $\sigma_i(V_\mu) = V_\nu$ for $\nu \stackrel{\text{def}}{=} r_{\alpha_i}(\mu)$ with $r_\alpha \in W$ the usual reflection.

Pf Follows from $\mathfrak{sl}_2$-repn theory since V_μ is $\overbrace{\text{fin-dim}}^{\text{by structure thm for standard cyclic modules}}$ S_i-submod.

Claim (7) If $\mu \in \Pi(V) = \Pi(\lambda)$ and $w \in W$ then $w(\mu) \in \Pi(\lambda)$ and $\dim V_{w(\mu)} = \dim V_\mu$

Pf Immediate from Claim (6) as $W = \langle r_{\alpha_i} \mid \alpha_i \in \Delta \rangle$ $\square$

Claim (8) $\Pi(\lambda)$ is finite

Pf $\Pi(\lambda)$ is a subset of the set of W-conjugates of all dominant integral $\mu \in H^*$ with $\mu < \lambda$ by prev claim and structure thm of standard cyclic modules. Results in chapter 13 of textbook imply this set is finite. $\square$

Claim (9) $\dim V < \infty$ since $\Pi(V) = \Pi(\lambda)$ is finite and each $\mu \in \Pi(\lambda)$ has $\dim V_\mu < \infty$ $\square$

$\square$

Multiplicity formula

Fix $\lambda \in \Lambda^+$. Then $V(\lambda)$ is fin. dim. irreducible.

For $\mu \in H^*$ let $m_\lambda(\mu) \overset{\text{def}}{=} \dim V(\lambda)_\mu \in \mathbb{Z}_{\geq 0}$

This is zero if $\mu \notin \Pi(\lambda)$. Call $m_\lambda(\mu)$ the multiplicity of μ in $V(\lambda)$.

If $\mu \in H^*$ and $\mu \notin \Lambda$ then $\mu \notin \Pi(\lambda)$ so $m_\lambda(\mu) = 0$.

Thm (Freudenthal's formula) If $\mu \in \Lambda$ and $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ then

$$\Big((\lambda+\delta, \lambda+\delta) - (\mu+\delta, \mu+\delta) \Big) m_\lambda(\mu) = 2 \sum_{\alpha \in \Phi^+} \sum_{i=1}^{\infty} m_\lambda(\mu + i\alpha)(\mu + i\alpha, \alpha)$$

and this formula provides an effective algorithm to compute $m_\lambda(\mu)$.

Key point (nontrivial, see §22 of textbook): if $\lambda \neq \mu$ then $\|\lambda+\delta\|^2 \neq \|\mu+\delta\|^2$

so can divide both sides by this number

minor point (trivial): $\boxed{m_\lambda(\lambda) = 1}$

Formal characters Want to assign to each fin. dim. L-module a vector (similar to character of a group repn) that identifies its isomorphism class.

Notation let $\mathbb{Z}[\Lambda]$ be the free $\mathbb{Z}$-module with basis given by symbols $\{e^\lambda \mid \lambda \in \Lambda\}$ and make this additive group into a ring by setting $e^\lambda e^\mu = e^{\lambda+\mu}$. Here $\Lambda \subset H^*$ is the infinite set of integral weights, including $0 \in \Lambda$.

Def If $\lambda \in \Lambda^+$ then the formal character of $V \cong V(\lambda)$ is

$$ch_V = ch_\lambda \overset{\text{def}}{=} \sum_{\mu \in \Pi(\lambda)} m_\lambda(\mu) e^\mu \in \mathbb{Z}[\Lambda].$$

If V is arb. finite dim. L-module then V has unique decomp. $V \cong V(\lambda_1) \oplus V(\lambda_2) \oplus \dots \oplus V(\lambda_k)$ with each $\lambda_i \in \Lambda^+$ and we set $ch_V = \sum_{i=1}^{k} ch_{\lambda_i}$

Ex If $L = \mathfrak{sl}_2(\mathbb{F})$ then $ch_\lambda = e^\lambda + e^{\lambda-\alpha} + e^{\lambda-2\alpha} + \dots + e^{\lambda - m\alpha}$

where $m = \langle \lambda, \alpha \rangle$ [Here $\alpha = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$, $m = \lambda_1 - \lambda_2$]

Weyl group W acts on $\mathbb{Z}[\Lambda]$ by

$$w \cdot \left(\sum_{\mu \in \Lambda} c_\mu e^\mu \right) = \sum_{\mu \in \Lambda} c_\mu e^{w(\mu)} \quad \text{where } c_\mu \in \mathbb{Z}$$

Cor ch_V is fixed by every $w \in W$. **Pf** $m_\lambda(\mu) = m_\lambda(w(\mu))$ $\forall w \in W$.

Prop If $f \in \mathbb{Z}[\Lambda]$ is fixed by all $w \in W$ then f has unique expansion as a finite linear combination of formal characters ch_λ for $\lambda \in \Lambda^+$.

Pf idea: write $f = \sum_{\lambda \in \Lambda} c_\lambda e^\lambda$ with $c_\lambda \in \mathbb{Z}$

all but finitely many c_λ's must be zero. Find a maximal $\lambda \in \Lambda^+$ with $c_\lambda \neq 0$, form $g = f - c_\lambda \mathrm{ch}_\lambda$, and argue that you may conclude by induction that g has desired expansion. $\square$

↳ need more to deduce uniqueness (exercise)

Prop Suppose V and W are both finite-dim. L-modules. Then $\mathrm{ch}_{V \otimes W} = \mathrm{ch}_V \, \mathrm{ch}_W$.

[Recall how $V \otimes W$ is an L-module:

$$X \cdot (v \otimes w) = Xv \otimes w + v \otimes Xw \quad \text{for } \begin{matrix} v \in V \\ w \in W \\ x \in L \end{matrix}$$
]

Pf Straightforward exercise. $\square$