FINAL EXAMINATION SOLUTIONS – MATH 2121, FALL 2021.

Problem 1. (30 points) This question has six parts.

(a) Find the general solution to the linear system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 0 \\ x_2 + x_3 + x_4 = 3 \\ x_3 + x_5 = 2. \end{cases}$$

Solution to part (a):

The augmented matrix of this linear system is

$$A = \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{array}\right].$$

Its reduced echelon form is

$$\mathsf{RREF}(A) = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 \end{array}\right].$$

This tells us that if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ is a solution then x_4 and x_5 are free variables while

$$x_1 = -3 - x_5, \quad x_2 = 1 - x_4 + x_5, \quad \text{and} \quad x_3 = 2 - x_5.$$

Therefore the general solution has the form

$$\boxed{x = \begin{bmatrix} -3-b \\ 1-a+b \\ 2-b \\ a \\ b \end{bmatrix} \quad \text{where } a, b \in \mathbb{R} \text{ are arbitrary.}}$$

(b) Find the standard matrix of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}$ with

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

Solution to part (b):

Since

$$T(x) = 2x_1 + 2x_3 + x_4 = \begin{bmatrix} 2 & 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

the standard matrix of T is $\boxed{\begin{bmatrix} 2 & 0 & 2 & 1 \end{bmatrix}}$.

(c) Find the value of h that makes the rank of the matrix

$$\begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 0 \\ 2 & 1 & h \\ 1 & 2 & 0 \end{bmatrix}$$

as small as possible.

Solution to part (c):

The second column is not a scalar multiple of the first, so the rank of the matrix is at least 2. The rank is exactly 2 if and only if the third column is a linear combination of the first two columns. There are real numbers $a, b \in \mathbb{R}$ such that

$$a\begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2a \\ a+2b \\ 2a+b \\ a+2b \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ h \\ 0 \end{bmatrix}$$

if and only if

$$a = 1 \quad \text{and} \quad b = -a/2 = -1/2 \quad \text{and} \quad h = 2a + b = 2 - 1/2 = 3/2,$$

so the answer is $\boxed{h = 3/2}$.

(d) Find all 2×3 matrices A that are in **reduced echelon form** and satisfy

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solution to part (d):

There was typo in this question in the printed exam, which referred to "3×2 matrices" instead of 2×3 matrices. In its original form, the question wouldn't really make sense and a reasonable answer would be $\boxed{\text{no such } A \text{ matrices exist.}}$

However, this mistake was announced in the main lecture venue and we only saw a few cases of exams which did not take into account the updated wording of the question. With this correction, the answer is

$$\boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & a & -a-1 \\ 0 & 0 & 0 \end{bmatrix} \text{ for all } a \in \mathbb{R}, \quad \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}}.$$

(e) Suppose $a, b, c, d, e \in \mathbb{R}$ are such that $ad - bc = 1$ and $e \neq 0$.
Compute the inverse of

$$A = \begin{bmatrix} 0 & a & b \\ 0 & c & d \\ e & 0 & 0 \end{bmatrix}.$$

Solution to part (e):

One can check by multiplying the matrices that $\boxed{A^{-1} = \begin{bmatrix} 0 & 0 & 1/e \\ d & -b & 0 \\ -c & a & 0 \end{bmatrix}}$.

(f) Suppose A is a 3×3 matrix with all real entries. The complex number $\lambda = 2 + 3i$ is an eigenvalue of A and the trace of A is $\mathrm{tr}(A) = 7$. What is the determinant of A?

Solution to part (f):

The trace is the sum of the eigenvalues and the determinant is the product of the eigenvalues (repeated with multiplicity). Since

$$\lambda_1 = 2 + 3i$$

is one eigenvalue it follows that

$$\lambda_2 = 2 - 3i$$

is another eigenvalue and so

$$\lambda_3 = \mathrm{tr}(A) - \lambda_1 - \lambda_2 = 7 - (2 + 3i) - (2 - 3i) = 7 - 4 = 3$$

is a third eigenvalue. Hence

$$\det(A) = \lambda_1 \lambda_2 \lambda_3 = (2 + 3i)(2 - 3i)3 = (4 - 9i^2)3 = (4 + 9)3 = 13 \times 3 = \boxed{39}.$$

Problem 2. (10 points) Do there exist two linearly independent vectors in $\mathbb{R}^4$ that are orthogonal to all three of the vectors

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 2 \\ 5 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ -5 \\ -2 \\ -7 \end{bmatrix}?$$

Find two such vectors if they exist, and otherwise explain why there are no such linearly independent vectors.

Solution:

The vectors in $\mathbb{R}^4$ orthogonal to the given vectors make up the null space of

$$\begin{bmatrix} 1 & -2 & 1 & 2 \\ 1 & -1 & 2 & 5 \\ 1 & -5 & -2 & -7 \end{bmatrix}.$$

The reduced echelon form of this matrix is

$$\begin{bmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

so the null space is 2-dimensional with a basis given by $\boxed{\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -8 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$.

It's a good idea to check that these vectors are in fact orthogonal to the three given vectors.

Problem 3. (10 points) This problem has two parts.

Suppose A is a 3×3 matrix such that

$$A \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix}, \quad A \begin{bmatrix} 12 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

(a) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Solution to part (a):

The columns of P should be linearly independent eigenvectors for A and the diagonal entries of D should be the corresponding eigenvalues. The given information provides exactly these eigenvectors and eigenvalues. There are multiple correct answers to this question but one would be

$$\boxed{P = \begin{bmatrix} 1 & 3 & 1 \\ -4 & 2 & -1 \\ 5 & 1 & -1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}}.$$

(b) Determine if $\lim_{n\to\infty} A^n$ exists and compute its value if it does exist.

Explain how you found your answer to receive full credit.

Solution to part (b):

Since $A^n = PD^nP^{-1} = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/4^n & 0 \\ 0 & 0 & 1/2^n \end{bmatrix} P^{-1}$ it follows that

$$\lim_{n\to\infty} A^n = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 3 & 1 \\ -4 & 2 & -1 \\ 5 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & 1 \\ -4 & 2 & -1 \\ 5 & 1 & -1 \end{bmatrix}^{-1}.$$

This answer is not completely explicit but would be enough to get almost full credit on this question. To fully compute the answer we need to find the inverse of P. A shortcut for this is to observe that the columns of P are orthogonal, so

$$P^\top P = \begin{bmatrix} 42 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Thus

$$P^{-1} = \begin{bmatrix} 42 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 3 \end{bmatrix}^{-1} P^\top = \begin{bmatrix} 1/42 & 0 & 0 \\ 0 & 1/14 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & -4 & 5 \\ 3 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix}.$$

Without multiplying everything out, we can compute

$$\begin{aligned} P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^{-1} &= P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/42 & 0 & 0 \\ 0 & 1/14 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} P^\top \\ &= P \begin{bmatrix} 1/42 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^\top \\ &= \frac{1}{42} P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} P^\top \\ &= \frac{1}{42} \begin{bmatrix} 1 & 0 & 0 \\ -4 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -4 & 5 \\ 3 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} \\ &= \frac{1}{42} \begin{bmatrix} 1 & -4 & 5 \\ -4 & 16 & -20 \\ 5 & -20 & 25 \end{bmatrix} \end{aligned}$$

and so

$$\boxed{\lim_{n\to\infty} A^n = \frac{1}{42} \begin{bmatrix} 1 & -4 & 5 \\ -4 & 16 & -20 \\ 5 & -20 & 25 \end{bmatrix}}.$$

Problem 4. (20 points) This problem has four parts.

Suppose A is a 3×3 matrix that has exactly two distinct (complex) eigenvalues given by -1 and 2, and that has all three of the following vectors as eigenvectors:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

(a) Can the matrix A be non-diagonalizable? If this is possible then give an example of such a matrix A, and otherwise explain why it is impossible.

Solution to part (a):

$\boxed{\text{No}}$. Any such matrix A has 3 linearly independent eigenvectors so is diagonalizable.

(b) Can the matrix A be non-invertible? If this is possible then give an example of such a matrix A, and otherwise explain why it is impossible.

Solution to part (b):

$\boxed{\text{No}}$. A matrix is non-invertible if and only if it has 0 are an eigenvalue, but A only has -1 and 2 as eigenvalues.

(c) Can the matrix A be orthogonal? (That is, can it hold that A is invertible with $A^{-1} = A^\top$?) If this is possible then give an example of such a matrix A, and otherwise explain why it is impossible.

Solution to part (c):

$\boxed{\text{No}}$. Orthogonal matrices define length-preserving linear transformations, so can only have ± 1 as eigenvalues. But A has 2 as an eigenvalue.

(d) Continue to suppose that A is a 3×3 matrix that has exactly two distinct (complex) eigenvalues given by -1 and 2, and that has all three of the following vectors as eigenvectors:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Can the matrix A be symmetric? (That is, can it hold that $A = A^\top$?) If this is possible then give an example of such a matrix A, and otherwise explain why it is impossible.

Solution to part (d):

$\boxed{\text{Yes}}$. Suppose $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ are in the -1-eigenspace of A while $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is in the 2-eigenspace. Then

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

is an orthonormal basis of $\mathbb{R}^3$ consisting of eigenvectors of A (with eigenvalues -1, -1, and 2 respectively), so the matrix

$$\boxed{A = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}}$$

is symmetric with the desired properties. This kind of answer is explicit enough to receive full credit on this problem.

Problem 5. (10 points) This question has two parts.

Consider the plane $P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 3x - y + 6z = 0 \right\}$ in $\mathbb{R}^3$.

(a) The subspace P is 2-dimensional. Find an orthogonal basis for P.

Solution to part (a):

Two linearly independent vectors in P are $\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix}$. The Gram-Schmidt process converts the second vector to

$$\begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix} - \frac{18}{10} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 1 \end{bmatrix} - \begin{bmatrix} 9/5 \\ 27/5 \\ 0 \end{bmatrix} = \begin{bmatrix} -9/5 \\ 3/5 \\ 1 \end{bmatrix}.$$

After rescaling this vector, we get an orthogonal basis for P given by

$$\boxed{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix}}.$$

(b) Find the vector in P that is closest to $v = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Solution to part (b):

The desired vector is the projection of v onto P which is

$$\frac{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + \frac{\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix}}{\begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix} \bullet \begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix}} \begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix}$$

which we can rewrite as

$$\begin{aligned} \frac{9}{10} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + \frac{-16}{115} \begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix} &= \frac{1}{230} \left(207 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} - 32 \begin{bmatrix} -9 \\ 3 \\ 5 \end{bmatrix} \right) \\ &= \frac{1}{230} \begin{bmatrix} 495 \\ 525 \\ -160 \end{bmatrix} \\ &= \boxed{\frac{1}{46} \begin{bmatrix} 99 \\ 105 \\ -32 \end{bmatrix}}. \end{aligned}$$

Problem 6. (15 points) This question has three parts.

(a) Suppose $A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 2 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Does the equation $Ax = b$ have an exact solution?

Find a solution or explain why none exists.

Solution to part (a):

The augmented matrix of the system $Ax = b$ is

$$\left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -1 & 1 \\ 2 & 2 & 2 \end{array}\right]$$

which has reduced echelon form

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right].$$

Since the last column is a pivot column, the original linear system is inconsistent so has no exact solution.

(b) Again suppose $A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 2 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Does the equation $Ax = b$ have a least-squares solution?

Find a solution or explain why none exists.

Solution to part (b):

The least-squares solutions to $Ax = b$ are the solutions to $A^\top Ax = A^\top b$. Since

$$A^\top A = \begin{bmatrix} 5 & 7 \\ 7 & 14 \end{bmatrix} \quad \text{and} \quad A^\top b = \begin{bmatrix} 6 \\ 9 \end{bmatrix}$$

the augmented matrix of $A^\top Ax = A^\top b$ is

$$\left[\begin{array}{cc|c} 5 & 7 & 6 \\ 7 & 14 & 9 \end{array}\right]$$

which has reduced echelon form

$$\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1/7 \end{array}\right].$$

Hence the unique least-squares solution to $Ax = b$ is $\boxed{x = \begin{bmatrix} 1 \\ 1/7 \end{bmatrix}}$.

(c) Suppose A is an $m \times n$ matrix and $b \in \mathbb{R}^m$.

Indicate which of the following are TRUE or FALSE.

You do not need to provide any justification for your answers.

Correct answers will receive 1 point, blank answers will recieve 0 points, and incorrect answers will lose 1 point.

1. If $x \in \mathbb{R}^n$ has $A^\top Ax = A^\top b$ then it always holds that $Ax = b$.

   TRUE **[FALSE]**

2. If $x \in \mathbb{R}^n$ has $Ax = b$ then it always holds that $A^\top Ax = A^\top b$.

   [TRUE] FALSE

3. If the equation $Ax = b$ has no solution then $A^\top Ax = A^\top b$ might also have no solution.

   TRUE **[FALSE]**

4. If the equation $Ax = b$ has a unique solution then $A^\top Ax = A^\top b$ also has a unique solution.

   [TRUE] FALSE

5. If the equation $A^\top Ax = A^\top b$ has a unique solution x then $Ax = b$ has at most one solution.

   [TRUE] FALSE

Problem 7. (10 points)

Define $\mathbb{R}^{3\times 3}$ to be the set of all 3×3 matrices with all real entries.

The set $\mathbb{R}^{3\times 3}$ is a vector space. Let

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$$

and define $T : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ by the formula $T(A) = JAJ$. This is a linear function.

Find all real numbers $\lambda \in \mathbb{R}$ such that $T(A) = \lambda A$ for some $0 \neq A \in \mathbb{R}^{3\times 3}$. For each of these eigenvalues λ find a basis for the subspace $\{A \in \mathbb{R}^{3\times 3} : T(A) = \lambda A\}$.

Solution :

First check that

$$T\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = \begin{bmatrix} i & h & g \\ f & e & d \\ c & b & a \end{bmatrix}.$$

Thus $T(T(A)) = A$ so if $T(A) = \lambda A$ then $\lambda^2 A = \lambda T(A) = T(T(A)) = A$. This means that the only possible eigenvalues of T are $\lambda = 1$ or $\lambda = -1$. Both numbers are in fact eigenvalues. A basis for the 1-eigenspace is

$$\boxed{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}.$$

A basis for the -1-eigenspace is

$$\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}.$$

Problem 8. (15 points) This question has three parts.

(a) Compute the singular values of the matrix $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$.

Solution to part (a):

The singular values of A are the square roots of the eigenvalues of

$$A^\top A = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}.$$

We have

$$\det(A^\top A - xI) = \det \begin{bmatrix} 9-x & -9 \\ -9 & 9-x \end{bmatrix} = (9-x)^2 - 9^2 = (-x)(18-x)$$

so the eigenvalues of $A^\top A$ are $\lambda = 0$ and $\lambda = 18$. Hence the singular values are

$$\boxed{\sigma_1 = \sqrt{18} > \sigma_2 = 0}.$$

(b) Suppose A is a 2×2 matrix with a singular value decomposition

$$A = U\Sigma V^T$$

where U and V are orthogonal 2×2 matrices and

$$\Sigma = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}.$$

The first column of U is the vector $\begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$.

Draw a picture of the region in $\mathbb{R}^2$ given by

$$\left\{ Ax : x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \text{ is a vector with } x_1^2 + x_2^2 \leq 1 \right\}.$$

Make your picture as detailed as possible to receive full credit.

Solution to part (b):

The correct answer is a picture of a solid ellipse centered at the origin with radius vectors $\pm \begin{bmatrix} -8 \\ 6 \end{bmatrix}$ and $\pm \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

(c) Find an orthonormal basis of $\mathbb{R}^3$ that contains the vector $\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$.

Solution to part (c):

A basis for the orthogonal complement of the span of this vector is

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Performing the Gram-Schmidt process converts the second vector to

$$\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{-4}{5}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}$$

so an orthogonal basis of our original orthogonal complement is

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}.$$

Converting these to unit vectors gives the desired orthonormal basis:

$$\boxed{\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}, \quad \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{45}}\begin{bmatrix} -2 \\ 4 \\ 5 \end{bmatrix}}.$$