FINAL EXAMINATION - MATH 2121, FALL 2017.

Name:

ID#:

Email:

Lecture & Tutorial:

Problem #	Max points possible	Actual score
1	15	
2	15	
3	10	
4	15	
5	15	
6	15	
7	10	
8	10	
9	15	
Total	120	

You have **180 minutes** to complete this exam.

No books, notes, or electronic devices can be used on the test.

Clearly label your answers by putting them in a box.

Partial credit can be given on some problems if you show your work. Good luck!

Problem 1. (3 + 3 + 3 + 3 + 3 = 15 points) Write complete, precise definitions of the following italicised terms.

(1) a *linear transformation* T from a vector space V to a vector space W.

(2) the *span* of a finite set of vectors $v_1, v_2, \ldots, v_n$ in a vector space.

(3) a *linearly independent* set of vectors $v_1, v_2, \ldots, v_n$ in a vector space.

(4) a *subspace* W of a vector space V.

(5) a *basis* for a vector space V.

Problem 2. (15 points) In the following statements, A, B, C, etc., are matrices (with all real entries), and $u, v, w, x,$, etc., are vectors in $\mathbb{R}^n$, unless otherwise noted.

Indicate which of the following is TRUE or FALSE.

One point will be given for each correct answer (no penalty for incorrect answers).

(1) Any system of n linear equations in n variables has at least n solutions.

TRUE FALSE

(2) If a linear system $Ax = b$ has more than one solution, then so does $Ax = 0$.

TRUE FALSE

(3) If A and B are $n \times n$ matrices with $AB = 0$, then $A = 0$ or $B = 0$.

TRUE FALSE

(4) If $AB = BA$ and A is invertible, then $A^{-1}B = BA^{-1}$.

TRUE FALSE

(5) If A is a square matrix, then $\det(-A) = -\det A$.

TRUE FALSE

(6) If A is a nonzero matrix then $\det A^T A > 0$.

TRUE FALSE

(7) If A is $m \times n$ and the transformation $x \mapsto Ax$ is onto, then $\operatorname{rank}(A) = m$.

TRUE FALSE

(8) If V is a vector space and $S \subset V$ is a subset whose span is V, then some subset of S is a basis of V.

TRUE FALSE

(9) If A is square and contains a row of zeros, then 0 is an eigenvalue of A.

TRUE FALSE

(10) Each eigenvector of a square matrix A is also an eigenvector of A^2.

TRUE FALSE

(11) If A is diagonalisable, then the columns of A are linearly independent.

TRUE FALSE

(12) Every 2×2 matrix (with all real entries) has an eigenvector in $\mathbb{R}^2$.

TRUE FALSE

(13) Every 3×3 matrix (with all real entries) has an eigenvector in $\mathbb{R}^3$.

TRUE FALSE

(14) If $\|u - v\|^2 = \|u\|^2 + \|v\|^2$ then vectors $u, v \in \mathbb{R}^m$ are orthogonal.

TRUE FALSE

(15) If the columns of A are orthonormal then AA^T is an identity matrix.

TRUE FALSE

Problem 3. (5 + 5 = 10 points)

(a) Compute the determinant of

$$A = \begin{bmatrix} a & 0 & b & 0 \\ c & 0 & d & 0 \\ 0 & a & 0 & b \\ 0 & c & 0 & d \end{bmatrix}$$

where a, b, c, d are real numbers.

For full credit, express your answer in as simple a form as possible.

(b) Find a matrix M such that $M \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $M \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$.

Problem 4. (5 + 5 + 5 = 15 points) Let $\mathcal{V}$ be the vector space of 3×3 matrices.

Define $L : \mathcal{V} \to \mathcal{V}$ as the linear transformation $L(A) = A + A^T$.

(a) Find a basis for the subspace $\mathcal{N} = \{A \in \mathcal{V} : L(A) = 0\}$. What is $\dim \mathcal{N}$?

(b) Find a basis for the subspace $\mathcal{R} = \{L(A) : A \in \mathcal{V}\}$. What is $\dim \mathcal{R}$?

(c) Find two numbers $\lambda, \mu \in \mathbb{R}$ and two nonzero matrices $A, B \in \mathcal{V}$ such that

$$L(A) = \lambda A \quad \text{and} \quad L(B) = \mu B.$$

Problem 5. (3 + 4 + 4 + 4 = 15 points) Let

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this problem A refers to a 3×3 matrix with all real entries satisfying

$$(A - I)(A - 2I)(A - 3I) = 0.$$

(a) Does there exist a 3×3 matrix A with $(A - I)(A - 2I)(A - 3I) = 0$ which is not diagonal? If there does, produce an example. Otherwise, give a short explanation for why no such matrix exists.

(b) Does there exist a 3×3 matrix A with $(A - I)(A - 2I)(A - 3I) = 0$ which has exactly 2 distinct eigenvalues? If there does, produce an example. Otherwise, give a short explanation for why no such matrix exists.

(c) Does there exist a 3×3 matrix A with $(A - I)(A - 2I)(A - 3I) = 0$ which does not have any of the numbers 1, 2, or 3 as an eigenvalue? If there does, produce an example. Otherwise, give a short explanation for why no such matrix exists.

(d) Does there exist a 3×3 matrix A with $(A - I)(A - 2I)(A - 3I) = 0$ which is not diagonalisable? If there does, produce an example. Otherwise, give a short explanation for why no such matrix exists.

Problem 6. (4 + 7 + 4 = 15 points)

(a) Compute the distinct eigenvalues of the matrix $A = \begin{bmatrix} .4 & -.3 \\ .4 & 1.2 \end{bmatrix}$.

(b) Again let $A = \begin{bmatrix} .4 & -.3 \\ .4 & 1.2 \end{bmatrix}$.

Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

(c) Continue to let $A = \begin{bmatrix} .4 & -.3 \\ .4 & 1.2 \end{bmatrix}$.

Find real numbers a, b, c, d such that $\lim_{n\to\infty} A^n = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Problem 7. (5 + 5 = 10 points)

(a) Find an orthonormal basis for the subspace of vectors of the form

$$\begin{bmatrix} a+2b+3c \\ 2a+3b+4c \\ 3a+4b+5c \\ 4a+5b+6c \end{bmatrix}$$

where a, b, c are real numbers.

(b) Find the vector in $W = \mathbb{R}\text{-span}\{u, v\}$ which is closest to y where

$$y = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$

Problem 8. (10 points) Describe all least-squares solutions to the linear equation

$$Ax = b$$

where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 7 \\ 2 \\ 3 \\ 6 \\ 5 \\ 4 \end{bmatrix}.$$

Problem 9. (3 + 5 + 7 = 15 points) Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of $A^T A$.

(b) Find an orthonormal basis v_1, v_2 for $\mathbb{R}^2$ consisting of eigenvectors of $A^T A$.

(c) Find a singular value decomposition for A. In other words, find the singular values $\sigma_1 \geq \sigma_2$ of A and then express A as a product

$$A = U\Sigma V^T$$

where U and V are invertible matrices with

$$U^{-1} = U^T \quad \text{and} \quad V^{-1} = V^T \quad \text{and} \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}.$$