FINAL EXAMINATION - MATH 2121, FALL 2018.

Name:

Student ID:

Email:

Tutorial: T1A T1B T2A T2B T3A T3B

Problem #	Max points possible	Actual score
1	20	
2	15	
3	10	
4	15	
5	20	
6	10	
7	15	
8	15	
Total	120	

You have **180 minutes** to complete this exam.

No books, notes, or electronic devices can be used on the test.

Clearly label your answers by putting them in a box.

Partial credit can be given on some problems if you show your work. Good luck!

Problem 1. (4 + 4 + 4 + 4 + 4 = 20 points)

(a) State the definition of a *linear transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$.

(b) Suppose $v_1 = \begin{bmatrix} -3 \\ 0 \\ 6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix}$, and $w = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}$.

Determine if w is in the span of v_1, v_2, and v_3.

Justify your answer to receive full credit.

(c) State the definition of the *dimension* a subspace V of $\mathbb{R}^n$.

(d) Let $v_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ -2 \end{bmatrix}$, $v_4 = \begin{bmatrix} 9 \\ 8 \\ 4 \\ 2 \end{bmatrix}$, and $v_5 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

Determine if the vectors v_1, v_2, v_3, v_4, v_5 are linearly independent.

Justify your answer to receive full credit.

(e) Consider the matrix

$$A = \begin{bmatrix} 1 & 7 & 0 \\ -2 & -3 & -3 \end{bmatrix}.$$

Find a 3×2 matrix B such that

$$Au \bullet v = u \bullet Bv$$

for all $u \in \mathbb{R}^3$ and $v \in \mathbb{R}^2$, where $\bullet$ denotes the vector inner product.

Problem 2. (15 points) In the following statements, A, B, C, etc., are matrices (with all real entries), and b, u, v, w, x, etc., are vectors, unless otherwise noted.

Indicate which of the following is TRUE or FALSE.

One point will be given for each correct answer (no penalty for incorrect answers).

(1) Every linear system with fewer equations than variables has a solution.

TRUE FALSE

(2) If w is a linear combination of u and v in $\mathbb{R}^n$, then u is a linear combination of v and w.

TRUE FALSE

(3) If A is an $n \times n$ matrix and I is the $n \times n$ identity matrix and $A^m = 0$ for some positive integer m, then $I - A$ is invertible.

TRUE FALSE

(4) If A is a 2×2 matrix and $\det A = 0$, then one row of A is a scalar multiple of the other row.

TRUE FALSE

(5) If A and B are row equivalent $m \times n$ matrices and the columns of A span $\mathbb{R}^m$, then so do the columns of B.

TRUE FALSE

(6) If A is an $m \times n$ matrix and the linear system $Ax = b$ has more free variables than basic variables, then $\mathrm{rank}\, A < \frac{n}{2}$.

TRUE FALSE

(7) If A is an $m \times n$ matrix then the function $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(x) = Ax$ for $x \in \mathbb{R}^n$ is one-to-one only if $\mathrm{Nul} A = \{0\}$.

TRUE FALSE

(8) If A is a 3×3 matrix and at least 6 entries in A are zero, then A is not invertible.

TRUE FALSE

(9) Each eigenvector of an invertible square matrix A is an eigenvector of A^{-1}.

TRUE FALSE

(10) If A is an $n \times n$ matrix with fewer than n distinct eigenvalues, then A is not diagonalizable.

TRUE FALSE

(11) An $n \times n$ matrix can have n distinct eigenvalues and exactly $n - 1$ real eigenvalues.

TRUE FALSE

(12) If the columns of A are orthonormal then A has at least as many rows as columns.

TRUE FALSE

(13) If W is a subspace of $\mathbb{R}^n$ then $(W^\perp)^\perp = W$.

TRUE FALSE

(14) If A is symmetric then A has all real eigenvalues.

TRUE FALSE

(15) Matrices with only real entries that have non-real eigenvalues can have singular value decompositions involving matrices with only real entries.

TRUE FALSE

Problem 3. (5 + 5 = 10 points)

(a) Compute the inverse of the matrix

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}.$$

(b) Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 \\ 0 & 1 & 4 & 4 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 4 & 1 \end{bmatrix}.$$

Problem 4. (5 + 5 + 5 = 15 points) Let

$$\mathcal{V} = \left\{ \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} : a, b, c, d, e, f, g, h, i \in \mathbb{R} \right\}$$

be the vector space of 3×3 matrices.

Define $L : \mathcal{V} \to \mathcal{V}$ to be the linear transformation

$$L\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right) = \begin{bmatrix} a+e+i & b+f & c \\ d+h & a+e+i & b+f \\ g & d+h & a+e+i \end{bmatrix}.$$

(a) Find a basis for the subspace $\mathcal{R} = \{L(A) : A \in \mathcal{V}\}$. What is $\dim \mathcal{R}$?

(b) Find a basis for the subspace $\mathcal{N} = \{A \in \mathcal{V} : L(A) = 0\}$. What is $\dim \mathcal{N}$?

(c) A number $\lambda \in \mathbb{R}$ is an *eigenvalue* for L if there exists a nonzero matrix $A \in \mathcal{V}$ with $L(A) = \lambda A$, in which case say that A is an *eigenvector* for L.

Find the distinct eigenvalues λ for L. For each eigenvalue λ, provide a nonzero matrix $A \in \mathcal{V}$ with $L(A) = \lambda A$.

Problem 5. (5 + 10 + 5 = 20 points)

(a) Compute the distinct eigenvalues of the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & -1 & -2 \\ 3 & 3 & 4 \end{bmatrix}$.

(b) Again let $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & -1 & -2 \\ 3 & 3 & 4 \end{bmatrix}$.

For each eigenvalue λ of A, find a basis for the eigenspace $\mathrm{Nul}(A - \lambda I)$.

(c) Continue to let $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & -1 & -2 \\ 3 & 3 & 4 \end{bmatrix}$.

Determine if A is diagonalizable. If A is diagonalizable, then give an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

If A is not diagonalizable, give an explanation why.

Problem 6. (5 + 5 = 10 points) Consider the symmetric matrix

$$A = \begin{bmatrix} 9/2 & 11/2 \\ 11/2 & 9/2 \end{bmatrix}.$$

(a) Find a 2×2 orthogonal matrix U and a 2×2 diagonal matrix D such that

$$A = UDU^T = UDU^{-1}.$$

(b) Continue to let

$$A = \begin{bmatrix} 9/2 & 11/2 \\ 11/2 & 9/2 \end{bmatrix}.$$

Find exact formulas for the functions $a(n)$, $b(n)$, $c(n)$, $d(n)$ such that

$$A^n = \begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix}$$

for all positive integers n.

Problem 7. (5 + 5 + 5 = 15 points) Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

(a) Find an orthogonal basis for the column space of A.

(b) Find a least-squares solution to the linear system $Ax = b$ where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}.$$

(c) Again let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$.

Compute the orthogonal projection of b onto the **orthogonal complement** of the column space of A.

Problem 8. (5 + 10 = 15 points)

(a) Compute the singular values $\sigma_1 \geq \sigma_2$ of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) Find a singular value decomposition for

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

In other words, find a 4×4 invertible matrix U and a 2×2 invertible matrix V with $U^{-1} = U^T$ and $V^{-1} = V^T$ such that

$$A = U\Sigma V^T \quad \text{where} \quad \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$