**FINAL EXAMINATION – MATH 2121, FALL 2021.**

Name:

Student ID:

Email:

Section: L1 (online) L2 (in person)

Tutorial: T1A T1B T1C T2A T2B T2C

| Problem # | Points Possible | Score |
|---|---|---|
| 1 | 30 | |
| 2 | 10 | |
| 3 | 10 | |
| 4 | 20 | |
| 5 | 10 | |
| 6 | 15 | |
| 7 | 10 | |
| 8 | 15 | |
| Total | 120 | |

$\binom{3}{3}$

You have **180 minutes** to complete this exam.

**No books, notes, or electronic devices can be used on the test.**

It will help us to grade your solutions if you draw a $\boxed{\text{box}}$ around your final answers to each problem. If we cannot determine what your final is on a problem, you may lose points. Partial credit can be given on some problems. Good luck!

**Problem 1.** (30 points) This question has six parts.

(a) Find the general solution to the linear system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 0 \\ x_2 + x_3 + x_4 = 3 \\ x_3 + x_5 = 2. \end{cases}$$

**Solution to part (a):**

(b) Find the standard matrix of the linear transformation $T : \mathbb{R}^4 \to \mathbb{R}$ with

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

**Solution to part (b):**

(c) Find the value of $h$ that makes the rank of the matrix

$$\begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 0 \\ 2 & 1 & h \\ 1 & 2 & 0 \end{bmatrix}$$

as small as possible.

**Solution to part (c):**

(d) Find all $2 \times 3$ matrices $A$ that are in **reduced echelon form** and satisfy

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

**Solution to part (d):**

(e) Suppose $a, b, c, d, e \in \mathbb{R}$ are such that $ad - bc = 1$ and $e \neq 0$.
Compute the inverse of

$$A = \begin{bmatrix} 0 & a & b \\ 0 & c & d \\ e & 0 & 0 \end{bmatrix}.$$

**Solution to part (e):**

(f) Suppose $A$ is a $3 \times 3$ matrix with all real entries. The complex number $\lambda = 2 + 3i$ is an eigenvalue of $A$ and the trace of $A$ is $\operatorname{tr}(A) = 7$. What is the determinant of $A$?

**Solution to part (f):**

**Problem 2.** (10 points) Do there exist two linearly independent vectors in $\mathbb{R}^4$ that are orthogonal to all three of the vectors

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1 \\ 2 \\ 5 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ -5 \\ -2 \\ -7 \end{bmatrix}?$$

Find two such vectors if they exist, and otherwise explain why there are no such linearly independent vectors.

**Solution:**

**Problem 3.** (10 points) This problem has two parts.

Suppose $A$ is a $3 \times 3$ matrix such that

$$A \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 5 \end{bmatrix}, \quad A \begin{bmatrix} 12 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

(a) Find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$.

**Solution to part (a):**

(b) Determine if $\lim_{n\to\infty} A^n$ exists and compute its value if it does exist.

Explain how you found your answer to receive full credit.

**Solution to part (b):**

**Problem 4.** (20 points) This problem has four parts.

Suppose $A$ is a $3 \times 3$ matrix that has exactly two distinct (complex) eigenvalues given by $-1$ and $2$, and that has all three of the following vectors as eigenvectors:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

(a) Can the matrix $A$ be non-diagonalizable? If this is possible then give an example of such a matrix $A$, and otherwise explain why it is impossible.

**Solution to part (a):**

(b) Continue to suppose that $A$ is a $3 \times 3$ matrix that has exactly two distinct (complex) eigenvalues given by $-1$ and $2$, and that has all three of the following vectors as eigenvectors:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Can the matrix $A$ be non-invertible? If this is possible then give an example of such a matrix $A$, and otherwise explain why it is impossible.

**Solution to part (b):**

(c) Continue to suppose that $A$ is a $3 \times 3$ matrix that has exactly two distinct (complex) eigenvalues given by $-1$ and $2$, and that has all three of the following vectors as eigenvectors:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Can the matrix $A$ be orthogonal? (That is, can it hold that $A$ is invertible with $A^{-1} = A^\top$?) If this is possible then give an example of such a matrix $A$, and otherwise explain why it is impossible.

**Solution to part (c):**

(d) Continue to suppose that $A$ is a $3 \times 3$ matrix that has exactly two distinct (complex) eigenvalues given by $-1$ and $2$, and that has all three of the following vectors as eigenvectors:

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}.$$

Can the matrix $A$ be symmetric? (That is, can it hold that $A = A^\top$?) If this is possible then give an example of such a matrix $A$, and otherwise explain why it is impossible.

**Solution to part (d):**

**Problem 5.** (10 points) This question has two parts.

Consider the plane $P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 3x - y + 6z = 0 \right\}$ in $\mathbb{R}^3$.

(a) The subspace $P$ is 2-dimensional. Find an orthogonal basis for $P$.

**Solution to part (a):**

(b) Find the vector in $P$ that is closest to $v = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

**Solution to part (b):**

**Problem 6.** (15 points) This question has three parts.

(a) Suppose $A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 2 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Does the equation $Ax = b$ have an exact solution?

Find a solution or explain why none exists.

**Solution to part (a):**

(b) Again suppose $A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 2 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Does the equation $Ax = b$ have a least-squares solution?

Find a solution or explain why none exists.

**Solution to part (b):**

(c) Suppose $A$ is an $m \times n$ matrix and $b \in \mathbb{R}^m$.

Indicate which of the following are TRUE or FALSE.

You do not need to provide any justification for your answers.

**Correct answers will receive $1$ point, blank answers will recieve $0$ points, and incorrect answers will lose $1$ point.**

1. If $x \in \mathbb{R}^n$ has $A^\top Ax = A^\top b$ then it always holds that $Ax = b$.

   TRUE FALSE

2. If $x \in \mathbb{R}^n$ has $Ax = b$ then it always holds that $A^\top Ax = A^\top b$.

   TRUE FALSE

3. If the equation $Ax = b$ has no solution then $A^\top Ax = A^\top b$ might also have no solution.

   TRUE FALSE

4. If the equation $Ax = b$ has a unique solution then $A^\top Ax = A^\top b$ also has a unique solution.

   TRUE FALSE

5. If the equation $A^\top Ax = A^\top b$ has a unique solution $x$ then $Ax = b$ has at most one solution.

   TRUE FALSE

**Problem 7.** (10 points)

Define $\mathbb{R}^{3\times 3}$ to be the set of all $3 \times 3$ matrices with all real entries.

The set $\mathbb{R}^{3\times 3}$ is a vector space. Let

$$J = \left[\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right] \in \mathbb{R}^{3\times 3}$$

and define $T : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}$ by the formula $T(A) = JAJ$. This is a linear function.

Find all real numbers $\lambda \in \mathbb{R}$ such that $T(A) = \lambda A$ for some $0 \neq A \in \mathbb{R}^{3\times 3}$. For each of these eigenvalues $\lambda$ find a basis for the subspace $\{A \in \mathbb{R}^{3\times 3} : T(A) = \lambda A\}$.

**Solution :**

**Problem 8.** (15 points) This question has three parts.

(a) Compute the singular values of the matrix $A = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$.

**Solution to part (a):**

(b) Suppose $A$ is a $2 \times 2$ matrix with a singular value decomposition

$$A = U\Sigma V^T$$

where $U$ and $V$ are orthogonal $2 \times 2$ matrices and

$$\Sigma = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}.$$

The first column of $U$ is the vector $\begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$.

Draw a picture of the region in $\mathbb{R}^2$ given by

$$\left\{ Ax : x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \text{ is a vector with } x_1^2 + x_2^2 \leq 1 \right\}.$$

Make your picture as detailed as possible to receive full credit.

**Solution to part (b):**

(c) Find an orthonormal basis of $\mathbb{R}^3$ that contains the vector $\begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$.

**Solution to part (c):**