FINAL EXAMINATION – MATH 2121, FALL 2022

Name:

Student ID:

Email:

Tutorial: T1A T1B T1C T1D T2A T2B T2C T2D

Problem #	Points Possible	Score
1	20	
2	10	
3	20	
4	10	
5	15	
6	15	
7	15	
8	15	
Total	120	

You have **180 minutes** to complete this exam.

No books, notes, or electronic devices can be used during the test.

RECOMMENDED: It will help us to grade your solutions if you draw a box around your answers to computational questions. If we cannot determine what your answer is, you may lose points. Partial credit can be given on some problems.

Good luck!

Problem 1. (20 points) This question has five parts. Each part asks you to provide a definition and then give a short derivation of a related property.

(a) Give the definition of a *linear function* $T : V \to W$ from a vector space V to a vector space W.

Then explain why your definition implies $T(0) = 0$ if $T : V \to W$ is linear.

(b) Give the definition of the *span* of three vectors $u, v, w \in V$ in a vector space.

Then explain why your definition implies that u, v, and w are each contained in $\mathbb{R}$-span$\{u, v, w\}$.

(c) Define what it means for three vectors $u, v, w \in V$ in a vector space to be *linearly dependent*.

Then explain why your definition implies that u, v, w are linearly dependent if $v = 0$.

(d) Define what it means for a subset H to be a *subspace* of a vector space V.

Then explain why your definition implies that the null space

$$\mathsf{Nul}(A) = \{v \in \mathbb{R}^n : Av = 0\}$$

of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.

(e) Define what it means for a set of vectors to be a *basis* of a vector space V.

Then explain why your definition implies that $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^3$.

Problem 2. (10 points) Let $A = \begin{bmatrix} -1 & 3 & -9 \\ 2 & 4 & -2 \\ 3 & 2 & 5 \end{bmatrix}$.

Compute the **reduced echelon form** $\mathsf{RREF}(A)$ of A.

Then find a **basis for** $\mathsf{Col}(A)$ and a **basis for** $\mathsf{Nul}(A)$. What is the **rank** of A?

Show all steps in your calculations to receive full credit.

Solution:

Problem 3. (20 points) This problem has four parts.

Consider the $2n \times 2n$ matrix

$$A = \begin{bmatrix} a & & & & & & & b \\ & a & & & & & b & \\ & & \ddots & & & \iddots & & \\ & & & a & b & & & \\ & & & b & a & & & \\ & & \iddots & & & \ddots & & \\ & b & & & & & a & \\ b & & & & & & & a \end{bmatrix}$$

that has $a \in \mathbb{R}$ in all positions on the main diagonal, $b \in \mathbb{R}$ in all positions on the main anti-diagonal, and zeros in all other positions. For example, if $n = 3$ then

$$A = \begin{bmatrix} a & 0 & 0 & 0 & 0 & b \\ 0 & a & 0 & 0 & b & 0 \\ 0 & 0 & a & b & 0 & 0 \\ 0 & 0 & b & a & 0 & 0 \\ 0 & b & 0 & 0 & a & 0 \\ b & 0 & 0 & 0 & 0 & a \end{bmatrix}.$$

However, for this problem we consider n to be an unspecified positive integer.

(a) Compute the **determinant** of A.

Show all steps in your calculations to receive full credit.

Solution to part (a):

(b) Find all **eigenvalues** of A.

Solution to part (b):

(c) Find an orthogonal matrix U and a diagonal matrix D such that

$$A = UDU^\top = UDU^{-1}.$$

(Remember that an *orthogonal matrix* has orthonormal columns).

Solution to part (c):

(d) Find all values of $a, b \in \mathbb{R}$ such that A is invertible and compute A^{-1}.

Solution to part (d):

Problem 4. (10 points) This problem has two parts.

(a) Determine the values of the constants $a, b \in \mathbb{R}$ such that the linear system

$$\begin{cases} x_1 + ax_2 + 2x_3 & = 2 \\ 4x_1 - 8x_2 + 8x_3 & = b \end{cases}$$

has (1) a unique solution, (2) infinitely many solutions, or (3) no solution.

Find the general solution in terms of a and b in cases (1) and (2).

Solution to part (a):

(b) Suppose A is a 3×3 matrix with all real entries.

The complex number $\lambda = 3 - 2i$ is an eigenvalue of A and $\det(A) = 65$.

What is the trace of A?

Explain how you found your answer to receive full credit.

Solution to part (b):

Problem 5. (15 points) This problem has five parts.

(a) Give an example of a diagonal square matrix that is not invertible.

(b) Give an example of a diagonalizable square matrix that is not diagonal.

(c) Give an example of a triangular square matrix that is not diagonalizable.

(d) Give an example of an invertible square matrix that is not triangular.

(e) Give an example of an orthogonal 2×2 matrix that is not a rotation matrix.

Problem 6. (15 points) This question has three parts.

(a) Find the orthogonal projection of the vector $v = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}$ onto the subspace

$$H = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + y + z = 0 \right\}.$$

Show all steps in your calculations to receive full credit.

Solution to part (a):

(b) Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line of best fit for the data points $(x, y) = (-1, 0)$, $(0, 1)$, $(1, 2)$, $(2, 4)$.

Sketch a plot of the data points along with your line of best fit.

Solution to part (b):

(c) Find $x, y \in \mathbb{R}$ that minimize the distance between $\begin{bmatrix} 2x \\ 0 \\ 2x \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ y \\ 2y \\ y \end{bmatrix}$.

Solution to part (c):

Problem 7. (15 points)

Define $\mathbb{R}^{2\times 2}$ to be the set of all 2×2 matrices with all real entries.

The set $\mathbb{R}^{2\times 2}$ is a vector space. Define $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by the formula

$$T(A) = \left[\begin{array}{cc} 2 & 0 \\ 2 & 2 \end{array}\right] A \left[\begin{array}{cc} 2 & 1 \\ 2 & 1 \end{array}\right].$$

This is a linear function.

(a) Find a basis for the subspace $\mathsf{range}(T) = \{T(A) : A \in \mathbb{R}^{2\times 2}\}$.

Solution to part (a):

(b) Find a basis for the subspace $\mathsf{kernel}(T) = \left\{ A \in \mathbb{R}^{2\times 2} : T(A) = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array}\right] \right\}$.

Solution to part (b):

(c) Find all **nonzero** numbers $\lambda \in \mathbb{R}$ such that $T(A) = \lambda A$ for some nonzero matrix $A \in \mathbb{R}^{2\times 2}$. For each of these nonzero eigenvalues λ, compute a basis for the subspace $\{A \in \mathbb{R}^{2\times 2} : T(A) = \lambda A\}$.

Solution to part (c):

Problem 8. (15 points) This question has three parts.

(a) Compute the singular values of the matrix $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$.

Solution to part (a):

(b) Suppose $A = U\Sigma V^\top$ is a singular value decomposition with

$$U = \begin{bmatrix} u_1 & u_2 & u_3 \\ u_4 & u_5 & u_6 \\ u_7 & u_8 & u_9 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 71 & 0 & 0 \\ 0 & 31 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad V^\top = \begin{bmatrix} v_1 & v_2 & v_3 \\ v_4 & v_5 & v_6 \\ v_7 & v_8 & v_9 \end{bmatrix}.$$

Find a basis for $\mathsf{Col}(A)$ and a basis for $\mathsf{Nul}(A)$.

Solution to part (b):

(c) Let $\mathbb{D}^2$ be the set of vectors $v \in \mathbb{R}^2$ with $\|v\| = 1$.

Suppose A is a 2×2 matrix with

$$\min\{\|Av\| : v \in \mathbb{D}^2\} = 20 \quad \text{and} \quad \max\{\|Av\| : v \in \mathbb{D}^2\} = 22.$$

Assume that $A \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 100 \end{bmatrix}$.

Draw a picture of the region $\{Av : v \in \mathbb{D}^2\}$ in $\mathbb{R}^2$.

Then determine all possible values for A.

Solution to part (c):