MIDTERM SOLUTIONS – SECTION L1, MATH 2121, FALL 2023

Problem 1. (10 points)

Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & x & 0 \\ y & 0 & 4 \end{bmatrix}.$$

Then find all values of $x, y \in \mathbb{R}$ such that A is invertible and compute a formula for A^{-1}.

Solution.

The determinant is $\det A = 1(4x - 0) - 0 + 2(0 - xy) = 4x - 2xy = \boxed{2x(2-y)}$.

A is invertible when its determinant is nonzero, which happens when $\boxed{x \neq 0 \text{ and } y \neq 2}$.

For these values of x and y we have $\boxed{A^{-1} = \begin{bmatrix} \frac{4}{4-2y} & 0 & \frac{-2}{4-2y} \\ 0 & \frac{1}{x} & 0 \\ \frac{-y}{4-2y} & 0 & \frac{1}{4-2y} \end{bmatrix}}$.

Problem 2. (10 points)

Find all values of $a, b \in \mathbb{R}$ such that $\mathbb{R}$-span $\left\{ \begin{bmatrix} 3 \\ 2 \\ a \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$ does **not** contain $\begin{bmatrix} 3 \\ 1 \\ b \end{bmatrix}$.

Solution.

We want to find a and b such that the matrix

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 2 & 2 & 1 \\ 0 & -1 & a & b \end{array}\right]$$

has a pivot in the last column, as then it is the augmented matrix of an inconsistent linear system.

Notice that we rearranged the spanning vectors when forming A to make it easier to row reduce.

We row reduce A by

$$A = \left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 2 & 2 & 1 \\ 0 & -1 & a & b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 1 & 1 & .5 \\ 0 & -1 & a & b \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 3 \\ 0 & 1 & 1 & .5 \\ 0 & 0 & a+1 & b+.5 \end{array}\right].$$

The last matrix is in echelon form (though not reduced) so its pivot positions are the same as in A.

We see that the last column has a pivot precisely when $\boxed{a = -1 \text{ and } b \neq -1/2}$.

Problem 3. (10 points)

Determine the possibilities for RREF $\begin{bmatrix} x & 1 & 2 \\ 1 & y & 3 \end{bmatrix}$ if x and y are real numbers.

Clearly identify the values of x and y that give rise to each reduced echelon form.

Solution.

We start to row reduce the matrix as

$$\begin{bmatrix} x & 1 & 2 \\ 1 & y & 3 \end{bmatrix} \to \begin{bmatrix} 1 & y & 3 \\ x & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & y & 3 \\ 0 & 1-xy & 2-3x \end{bmatrix}.$$

If $\boxed{xy \neq 1}$ then we can continue row reducing

$$\begin{bmatrix} 1 & y & 3 \\ 0 & 1-xy & 2-3x \end{bmatrix} \to \begin{bmatrix} 1 & y & 3 \\ 0 & 1 & \frac{2-3x}{1-xy} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3-\frac{2-3x}{1-xy}y \\ 0 & 1 & \frac{2-3x}{1-xy} \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 & \frac{3-2y}{1-xy} \\ 0 & 1 & \frac{2-3x}{1-xy} \end{bmatrix}}$$

where the last matrix is in RREF form.

If $\boxed{xy = 1}$ and $\boxed{x \neq 2/3}$ then we can continue row reducing

$$\begin{bmatrix} 1 & y & 3 \\ 0 & 1-xy & 2-3x \end{bmatrix} = \begin{bmatrix} 1 & y & 3 \\ 0 & 0 & 2-3x \end{bmatrix} \to \begin{bmatrix} 1 & y & 3 \\ 0 & 0 & 1 \end{bmatrix} \to \boxed{\begin{bmatrix} 1 & y & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

where the last matrix is in RREF form.

If $xy = 1$ and $\boxed{x = 2/3}$ then $\boxed{y = 3/2}$ and

$$\begin{bmatrix} 1 & y & 3 \\ 0 & 1-xy & 2-3x \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 3/2 & 3 \\ 0 & 0 & 0 \end{bmatrix}}$$

is in RREF form.

Problem 4. (10 points)

This question has two parts:

(a) What is the definition of a **one-to-one linear** function $f : \mathbb{R}^n \to \mathbb{R}^m$?

(b) Find the standard matrix of the unique linear function $f : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$f\left(\left[\begin{array}{c} 2 \\ 3 \end{array}\right]\right) = \left[\begin{array}{c} 2 \\ 3 \end{array}\right] \quad \text{and} \quad f\left(\left[\begin{array}{c} 2 \\ 1 \end{array}\right]\right) = \left[\begin{array}{c} -2 \\ -1 \end{array}\right].$$

Is f one-to-one? Justify your answer.

Solution.

Solution.

(a) A one-to-one linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function with $f(cv) = cf(v)$ and $f(v+w) = f(v)+f(w)$ for all $c \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$, such that for each $y \in \mathbb{R}^m$ there is at most one $x \in \mathbb{R}^n$ with $f(x) = y$.

The last property could be rephrased as: the standard matrix of f has a pivot position in every column.

(b) If the standard matrix of f is A then $A\left[\begin{array}{cc} 2 & 2 \\ 3 & 1 \end{array}\right] = \left[\begin{array}{cc} 2 & -2 \\ 3 & -1 \end{array}\right]$ so

$$A = \left[\begin{array}{cc} 2 & -2 \\ 3 & -1 \end{array}\right]\left[\begin{array}{cc} 2 & 2 \\ 3 & 1 \end{array}\right]^{-1} = -\frac{1}{4}\left[\begin{array}{cc} 2 & -2 \\ 3 & -1 \end{array}\right]\left[\begin{array}{cc} 1 & -2 \\ -3 & 2 \end{array}\right] = -\frac{1}{4}\left[\begin{array}{cc} 8 & -8 \\ 6 & -8 \end{array}\right] = \boxed{\left[\begin{array}{cc} -2 & 2 \\ -3/2 & 2 \end{array}\right]}.$$

Since $\det A = -4 + 3 = -1 \neq 0$, the matrix A is invertible.

Therefore $\mathsf{RREF}(A) = I$ has a pivot in every column, so $\boxed{f \text{ is one-to-one}}$.

Problem 5. (10 points)

This question has two parts:

(a) Does there exist a 2×2 matrix A with $\mathrm{Col}\, A = \mathrm{Nul}\, A$?

Find an example or explain why none exists.

(b) Does there exist a 3×3 matrix A with $\mathrm{Col}\, A = \mathrm{Nul}\, A$?

Find an example or explain why none exists.

Solution.

(a) $\boxed{\text{Yes}}$. The matrix $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ has $\mathrm{Col}\, A = \mathrm{Nul}\, A = \mathbb{R}$-span $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

(b) $\boxed{\text{No}}$. If A is 3×3 then $\dim \mathrm{Col}\, A + \dim \mathrm{Nul}\, A = 3$ so $\dim \mathrm{Col}\, A$ and $\dim \mathrm{Nul}\, A$ cannot be both odd or both even, so $\dim \mathrm{Col}\, A \neq \dim \mathrm{Nul}\, A$ which means that $\mathrm{Col}\, A \neq \mathrm{Nul}\, A$.

Problem 6. (10 points)

Define the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & -1 \\ 2 & 0 & -4 & 1 & 0 \end{bmatrix}$. Find a **basis for** $\operatorname{Col} A$ and a **basis for** $\operatorname{Nul} A$.

Solution.

We row reduce A as

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & -1 \\ 2 & 0 & -4 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & -2 & -6 & -1 & -2 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} = \mathsf{RREF}(A).$$

Columns 1, 2, and 4 have pivots so a basis for $\operatorname{Col} A$ is $\boxed{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$.

The solutions to $Ax = 0$ are the same as $\mathsf{RREF}(A)x = 0$, which can be rewritten as

$$\begin{cases} x_1 - 2x_3 + 2x_5 = 0 \\ x_2 + 3x_3 + 3x_5 = 0 \\ x_4 - 4x_5 = 0. \end{cases}$$

This means that $x \in \operatorname{Nul} A$ if and only if

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_3 - 2x_5 \\ -3x_3 - 3x_5 \\ x_3 \\ 4x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ -3 \\ 0 \\ 4 \\ 1 \end{bmatrix}.$$

Therefore a basis for $\operatorname{Nul} A$ is $\boxed{\begin{bmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -3 \\ 0 \\ 4 \\ 1 \end{bmatrix}}$.