MIDTERM SOLUTIONS– SECTION L2, MATH 2121, FALL 2023

Problem 1. (10 points)

Find the general solution to the linear system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 & = 1 \\ x_1 + 2x_2 + 4x_3 + 2x_4 & = 0 \\ 2x_1 - 4x_3 + x_4 & = 0. \end{cases}$$

Solution.

The augmented matrix of the system is $A = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 & 0 \\ 2 & 0 & -4 & 1 & 0 \end{array}\right]$.

We row reduce this as

$$A = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 & 0 \\ 2 & 0 & -4 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & -2 & -6 & -1 & -2 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 5 \\ 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right] = \mathsf{RREF}(A).$$

The linear system with augmented matrix $\mathsf{RREF}(A)$ is

$$\begin{cases} x_1 - 2x_3 & = 2 \\ x_2 + 3x_3 & = 3 \\ x_4 & = -4. \end{cases}$$

This means the basic variables are $x_1 = 2 + 2x_3$, $x_2 = 3 - 3x_3$, and $x_4 = -4$ so the general solution is

$$\boxed{(x_1, x_2, x_3, x_4) = (2 + 2a, 3 - 3a, a, -4) \quad \text{for all } a \in \mathbb{R}}.$$

Problem 2. (10 points)

Determine the values of the constants a and b such that the matrix equation

$$\begin{bmatrix} a & 6 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ b \end{bmatrix}$$

has (1) a unique solution, (2) no solution, or (3) infinitely many solutions.

Solution.

The system has a unique solution if $\begin{bmatrix} a & 6 \\ -1 & 3 \end{bmatrix}$ is invertible.

This happens if $3a + 6 \neq 0$ or equivalently $a \neq -2$.

Assume $a = -2$. Then the augmented matrix of the matrix equation is $A = \left[\begin{array}{cc|c} -2 & 6 & 3 \\ -1 & 3 & b \end{array}\right]$.

This row reduces to

$$A = \left[\begin{array}{cc|c} -2 & 6 & 3 \\ -1 & 3 & b \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & -1.5 \\ -1 & 3 & b \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -3 & -1.5 \\ 0 & 0 & b - 1.5 \end{array}\right].$$

The last matrix is in echelon form so its pivot positions are the pivot positions of $\mathsf{RREF}(A)$.

There is no solution if the last column is a pivot. This happens when $b - 1.5 \neq 0$ or equivalently $b \neq 3/2$.

There are infinitely many solutions if the last column is not a pivot as then x_2 is a free variable.

This happens when $b - 1.5 = 0$ or equivalently $b = 3/2$.

The final answer is therefore

- $\boxed{\text{unique solution if } a \neq -2}$,
- $\boxed{\text{no solution if } a = -2 \text{ and } b \neq 3/2}$, and
- $\boxed{\text{infinitely many solutions if } a = -2 \text{ and } b = 3/2}$.

Problem 3. (10 points)

Let x be a real number and define

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 2 \\ 3 & 1 & x \end{bmatrix}.$$

Compute the **rank** and **determinant** of the matrix A. Your answer should depend on x.

Solution.

The determinant is $1(4x-2) - 3(2x-6) + 0 = 4x - 2 - 6x + 18 = 16 - 2x$. So $\boxed{\det A = 16 - 2x}$.

We row reduce A as

$$A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 2 \\ 3 & 1 & x \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 2 \\ 0 & -8 & x \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 \\ 0 & -2 & 2 \\ 0 & 0 & x-8 \end{bmatrix}.$$

The last matrix is in echelon form so its pivot positions are the pivot positions of $\mathsf{RREF}(A)$.

If $x \neq 8$ then every diagonal position is a pivot.

Then all three columns of A form a basis for the column space, and in this case $\text{rank}A = 3$.

If $x = 8$ then only the first diagonal positions are pivots.

Then just the first two columns of A are a basis for the column space, and in this case $\text{rank}A = 2$.

So the rank of A is $\boxed{\text{rank}A = \begin{cases} 2 & \text{if } x = 8 \\ 3 & \text{if } x \neq 8. \end{cases}}$

Problem 4. (10 points)

This question has two parts:

(a) What is the definition of an **onto linear** function $f : \mathbb{R}^n \to \mathbb{R}^m$?

(b) Find the standard matrix of the linear function $f : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$f\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}.$$

Is f onto? Justify your answer.

Solution.

(a) An onto linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function with $f(cv) = cf(v)$ and $f(v + w) = f(v) + f(w)$ for all $c \in \mathbb{R}$ and $v, w \in \mathbb{R}^n$, such that for each $y \in \mathbb{R}^m$ there is at least one $x \in \mathbb{R}^n$ with $f(x) = y$.

The last property could be rephrased as: the standard matrix of f has a pivot position in every row.

(b) The standard matrix of f is $A = \begin{bmatrix} f(e_1) & f(e_2) & f(e_3) & f(e_4) \end{bmatrix}$.

We have $f(e_1) = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$.

We have $f(e_2) = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \end{bmatrix}$.

We have $f(e_3) = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \end{bmatrix}$.

We have $f(e_4) = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \end{bmatrix}$.

The standard matrix is therefore $\boxed{A = \begin{bmatrix} 10 & 12 & 0 & 0 \\ 10 & 12 & 15 & 18 \end{bmatrix}}$.

This matrix has a pivot in every row since it is row equivalent to $\begin{bmatrix} 10 & 12 & 0 & 0 \\ 0 & 0 & 15 & 18 \end{bmatrix}$ which is in echelon form and has two nonzero rows. Therefore $\boxed{f \text{ is onto}}$.

Problem 5. (10 points)

Two subspaces of $\mathbb{R}^n$ are **disjoint** if the only vector they both contain is the zero vector.

This question has two parts:

(a) Does there exist a non-invertible 2×2 matrix A such that $\mathrm{Col}\,A$ and $\mathrm{Nul}\,A$ are disjoint?

Find an example or explain why none exists.

(b) Does there exist a non-invertible 3×3 matrix A such that $\mathrm{Col}\,A$ and $\mathrm{Nul}\,A$ are disjoint?

Find an example or explain why none exists.

Solution.

(a) $\boxed{\text{Yes}}$. The matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible and has $\mathrm{Col}\,A = \mathbb{R}\text{-span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$ and $\mathrm{Nul}\,A = \mathbb{R}\text{-span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ which are disjoint subspaces.

(b) $\boxed{\text{Yes}}$. The matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is not invertible and has $\mathrm{Col}\,A = \mathbb{R}\text{-span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$ and $\mathrm{Nul}\,A = \mathbb{R}\text{-span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ which are disjoint subspaces.

Problem 6. (10 points)

Let $v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$.

Compute $A = vv^\top + ww^\top$. Then find a **basis for** $\operatorname{Col} A$ and a **basis for** $\operatorname{Nul} A$.

Solution.

We have

$$vv^\top = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$ww^\top = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix}$$

so

$$\boxed{A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 1 & 4 & 7 & 10 \end{bmatrix}}.$$

This matrix row reduces to

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 1 & 4 & 7 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(A).$$

The first two columns have pivots so a basis for $\operatorname{Col} A$ is $\boxed{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}$.

The linear system $\mathsf{RREF}(A)x = 0$ can be rewritten (ignoring the trivial equations $0 = 0$) as

$$\begin{cases} x_1 - x_3 - 2x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{cases}$$

which means that $Ax = 0$ if and only if $x_1 = x_3 + 2x_4$ and $x_2 = -2x_3 - 3x_4$ or equivalently

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 \\ -2x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

so a basis for $\operatorname{Nul}(A)$ is $\boxed{\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$.