MIDTERM SOLUTIONS – MATH 2121, FALL 2021.

Problem 1. (10 points)

Suppose A is a 2×3 matrix whose columns span $\mathbb{R}^2$.

(a) Describe all matrices that could occur as the reduced echelon form of A. Be as specific as possible.

(b) Suppose further that $A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for some $a, b, c \in \mathbb{R}$ with $c \neq 0$.

Describe all matrices that could occur as the reduced echelon form of A.

Solution:

(a) The columns of A span $\mathbb{R}^2$ so A must have pivot positions in every row. Therefore the possibilities for the reduced echelon form of A are

$$\boxed{\begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \end{bmatrix}, \quad \begin{bmatrix} 1 & u & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

where $u, v \in \mathbb{R}$ are arbitrary real numbers.

(b) The linear systems $Ax = 0$ and $\mathsf{RREF}(A)x = 0$ have the same solutions because their augmented matrices are row equivalent. Therefore

$$\mathsf{RREF}(A) \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If $\mathsf{RREF}(A) = \begin{bmatrix} 1 & 0 & u \\ 0 & 1 & v \end{bmatrix}$ then

$$\mathsf{RREF}(A) \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + uc \\ b + vc \end{bmatrix}$$

which is the zero vector if and only if $u = -a/c$ and $v = -b/c$.

If $\mathsf{RREF}(A)$ is $\begin{bmatrix} 1 & u & 0 \\ 0 & 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then $\mathsf{RREF}(A) \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is nonzero since its second entry is $c \neq 0$, so $\mathsf{RREF}(A)$ cannot have this form.

Therefore we must have $\boxed{\mathsf{RREF}(A) = \begin{bmatrix} 1 & 0 & -a/c \\ 0 & 1 & -b/c \end{bmatrix}}$.

Problem 2. (15 points)

Suppose a and b are real numbers. Consider the lines

$$L_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = ax \right\} \quad \text{and} \quad L_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = bx \right\}.$$

(a) When is it impossible to express the vector $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ as a sum of two vectors, one on the line L_1 and one on the line L_2?

(b) When is there more than one way of expressing the vector $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ as a sum of two vectors, one on the line L_1 and one on the line L_2?

(c) When is there exactly one way of writing

$$\begin{bmatrix} 5 \\ 6 \end{bmatrix} = v + w$$

with $v \in L_1$ and $w \in L_2$? Find a formula for v and w in this case.

Solution:

(a) This happens if and only if $L_1 = L_2$ are the same line and $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ is not on this line. In other words, when $\boxed{a = b \neq \frac{6}{5}}$.

(b) This happens if and only if $L_1 = L_2$ are the same line and $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ is on this line. In other words, when $\boxed{a = b = \frac{6}{5}}$.

(c) This happens when L_1 and L_2 are distinct lines, that is, when $\boxed{a \neq b}$.

Expressing $\begin{bmatrix} 5 \\ 6 \end{bmatrix} = v + w$ where $v \in L_1$ and $w \in L_2$ means finding numbers $x_1, x_2 \in \mathbb{R}$ such that $\begin{bmatrix} x_1 \\ ax_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ bx_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

We can rewrite this as the matrix equation

$$(*) \qquad \begin{bmatrix} 1 & 1 \\ a & b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

which we solve by row reducing the augmented matrix

$$\begin{bmatrix} 1 & 1 & 5 \\ a & b & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 5 \\ 0 & b-a & 6-5a \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 5 \\ 0 & 1 & \frac{6-5a}{b-a} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 - \frac{6-5a}{b-a} \\ 0 & 1 & \frac{6-5a}{b-a} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \frac{5b-6}{b-a} \\ 0 & 1 & \frac{6-5a}{b-a} \end{bmatrix}.$$

Thus the unique solution to (*) has $x_1 = \frac{5b-6}{b-a}$ and $x_2 = \frac{6-5a}{b-a}$ which gives

$$\boxed{v = \frac{5b-6}{b-a} \begin{bmatrix} 1 \\ a \end{bmatrix} \quad \text{and} \quad w = \frac{6-5a}{b-a} \begin{bmatrix} 1 \\ b \end{bmatrix}.}$$

Problem 3. (10 points)

(a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates a vector counterclockwise by 45 degrees and then doubles its length. Find the standard matrix of T, that is, the matrix A such that $T(v) = Av$ for all $v \in \mathbb{R}^2$.

(b) Let M be a 2×2 rotation matrix not equal to the identity matrix.

Suppose $M^{-1} = M^5$ and $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

How many different vectors could be in the set

$$S = \{v,\ Mv,\ M^2v,\ M^3v,\ M^4v,\ M^5v\}?$$

For each possibility, draw a picture representing the vectors in S and compute the sum $v + Mv + M^2v + M^3v + M^4v + M^5v$.

Solution:

(a) The standard basis vector $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ rotated CCW by 45 degrees is $\begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$.

The standard basis vector $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ rotated CCW by 45 degrees is $\begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$.

Doubling the length of these vectors gives $\begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix}$ and $\begin{bmatrix} -\sqrt{2} \\ \sqrt{2} \end{bmatrix}$ which are the columns of the desired standard matrix

$$\boxed{A = \begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}.}$$

(b) Since $M^{-1} = M^5$, the matrix M^6 is the identity matrix. Thus if M rotates all vectors CCW by angle θ, then 6θ must be a multiple of 360 degrees. The angle θ cannot be zero since M is not the identity matrix. It follows that θ is either 60, 120, or 180 degrees, so the set S has either $\boxed{2,\ 3,\ \text{or } 6 \text{ elements}}$.

If θ is 60 degrees then S consists of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and fives copies of this vector rotated by 60, 120, 180, 240, and 300 degrees CCW.

If θ is 120 degrees then S consists of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and two copies of this vector rotated by 120 and 240 degrees CCW.

If θ is 180 degrees then S consists of just $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$.

If $w = v + Mv + M^2v + M^3v + M^4v + M^5v$ then $Mw = Mv + M^2v + M^3v + M^4v + M^5v + v = w$. But M rotates w by a nonzero angle, so it is only possible to have $Mw = w$ if $\boxed{w = 0}$.

Problem 4. (5 points)

Find the value(s) of $h \in \mathbb{R}$ for which the following vectors are linearly dependent:

$$\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} -6 \\ 8 \\ 7 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ -2 \\ h \end{bmatrix}.$$

Solution:

The given vectors are linearly dependent if and only if the matrix

$$\begin{bmatrix} 1 & -6 & 4 \\ -3 & 8 & -2 \\ 4 & 7 & h \end{bmatrix}$$

has fewer than three pivot columns. Row reducing this matrix gives

$$\begin{bmatrix} 1 & -6 & 4 \\ -3 & 8 & -2 \\ 4 & 7 & h \end{bmatrix} \to \begin{bmatrix} 1 & -6 & 4 \\ 0 & -10 & 10 \\ 4 & 7 & h \end{bmatrix} \to \begin{bmatrix} 1 & -6 & 4 \\ 0 & 1 & -1 \\ 4 & 7 & h \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 4 & 0 & h+7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & h+15 \end{bmatrix}.$$

Thus there are fewer than three pivot columns if and only if $\boxed{h = -15}$.

Problem 5. (15 points)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
Suppose H is a k-dimensional subspace of $\mathbb{R}^n$. Define the set

$$T(H) = \{T(v) : v \in H\}.$$

(a) Explain why $T(H)$ is a subspace of $\mathbb{R}^m$.

(b) If T is onto, then what are the possibilities for $\dim T(H)$?
Justify your answer, which should be in terms of k, m, and n.

(c) If T is one-to-one, then what are the possibilities for $\dim T(H)$?
Justify your answer, which should be in terms of k, m, and n.

Solution:

(a) Let A be the standard matrix of T and let B be a $n \times k$ matrix whose columns are a basis for H. Then $H = \operatorname{Col} B$ and $T(H) = \operatorname{Col}(AB)$. This is a subspace since column spaces are subspaces.

(b) Suppose $u_1, \dots, u_j, v_1, \dots, v_{k-j}$ is a basis for H where each $T(u_1) = \cdots = T(u_j) = 0$, and j is as large as possible. Then $T(v_1), \dots, T(v_{k-j})$ are linearly independent since if $c_1 T(v_1) + \cdots + c_{k-j} T(v_{k-j}) = 0$ then $T(c_1 v_1 + \cdots + c_{k-j} v_{k-j}) = 0$ which can only happen if $c_1 = \cdots = c_{k-j} = 0$ as otherwise we could set $u_{j+1} = c_1 v_1 + \cdots + c_{k-j} v_{k-j}$.

Since $T(H)$ is spanned by $T(v_1), \dots, T(v_{k-j})$, we have $\dim T(H) = k - j$. This is at most k, and also at most m since $T(H) \subseteq \mathbb{R}^m$.

The value of j is at most $\dim\{v \in \mathbb{R}^n : T(v) = 0\} = \dim \operatorname{Nul} A = n - \operatorname{rank} A$ and also at most k. If T is onto then $\operatorname{rank} A = m \leq n$ so $j \leq \min\{n - m, k\}$ and therefore $\boxed{\max\{k - (n - m), 0\} \leq \dim T(H) \leq \min\{k, m\}}$.

(c) If T is one-to-one then $k \leq n \leq m$ and $T(v) \neq 0$ if $v \neq 0$, so we must have $j = 0$ and $\boxed{\dim T(H) = k}$.

Problem 6. (10 points) Suppose $A = \begin{bmatrix} u & v & w & x & y & z \end{bmatrix}$ is a 4×6 matrix with columns $u, v, w, x, y, z \in \mathbb{R}^4$. The reduced echelon form of A is

$$\mathsf{RREF}(A) = \begin{bmatrix} 0 & 1 & 0 & 3 & 0 & -2 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the null space of A.
(b) Find a basis for the column space of A.

(a) We have $Ax = 0$ if and only if $\mathsf{RREF}(A)x = 0$, which holds if and only if

$$\begin{cases} x_2 + 3x_4 - 2x_6 = 0 \\ x_3 - x_4 = 0 \\ x_5 + x_6 = 0. \end{cases}$$

In other words any $x \in \mathrm{Nul}\, A$ must have the form

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} x_1 \\ -3x_4 + 2x_6 \\ x_4 \\ x_4 \\ -x_6 \\ x_6 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

This means that a basis for $\mathrm{Nul}\, A$ is

$$\boxed{\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}}.$$

(b) A basis for $\mathrm{Col}\, A$ is given by the pivot columns in A, which are $\boxed{\{v, w, y\}}$.

Problem 7. (15 points)

Let $n \geq 2$ be a positive integer. Suppose A is the $n \times n$ matrix with 0's on the main diagonal and 1's everywhere else. For example, if $n = 4$ then we would have

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Let I be the $n \times n$ identity matrix.

(a) Find numbers b and c such that $A^2 = bI + cA$. (These will depend on n.)
(b) Compute a formula for the inverse of A. Be as specific as possible.
(c) Compute a formula for $\det(A)$.

Solution:

(a) Multiplying row i by column i of A results in a sum of n numbers, one of which is zero and the rest of which are one. Multiplying row i by column j of A when $i \neq j$ results in a sum of n numbers, two of which are zero and the rest of which are one. Therefore A^2 has $n-1$ in all diagonal positions and $n-2$ in all other positions, meaning that $A^2 = (n-1)I + (n-2)A$ so

$$\boxed{b = n-1 \text{ and } c = n-2}.$$

(b) By part (a) we have $A(A + (2-n)I) = A^2 + (2-n)A = (n-1)I$ so

$$A\left(\tfrac{1}{n-1}A + \tfrac{2-n}{n-1}I\right) = I.$$

Since A is square this means that $\boxed{A^{-1} = \frac{1}{n-1}A + \frac{2-n}{n-1}I}$.

(c) Replace the first row of A by the sum of all of its rows. This gives a row equivalent matrix with the same determinant whose first row is

$$\begin{bmatrix} n-1 & n-1 & \dots & n-1 \end{bmatrix}$$

and whose other rows are the same as in A. Next subtract $\frac{1}{n-1}$ times the new first row from all other rows. This gives another matrix which is row equivalent to A with the same determinant. This new matrix looks like

$$\begin{bmatrix} n-1 & n-1 & n-1 & \cdots & n-1 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \end{bmatrix}.$$

The matrix has $n-1$ in all entries in the first row, -1 in all diagonal entries after the first row, and zero in all other entries. It is triangular so its determinant is the product of its diagonal entries. This product is

$$\boxed{(n-1)(-1)^{n-1} = \det A}.$$