MIDTERM SOLUTIONS – MATH 2121, FALL 2022

Problem 1. (3 + 3 + 3 + 3 + 8 = 20 points)

This problem has five parts.

(a) There are sixty-four 3×2 matrices whose entries are each zero or one. An example of such a matrix is $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. List all of the 3×2 matrices whose entries are each zero or one that are in reduced echelon form.

Solution:

There are 5 such matrices:

$$\boxed{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}.$$

(b) Suppose $v_1, v_2, v_3, v_4, v_5 \in \mathbb{R}^4$ are the columns of the 4×5 matrix

$$A = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \end{bmatrix}.$$

The reduced echelon form of A is

$$\mathsf{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

What is the reduced echelon form of the matrix $B = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$?

Solution:

Since B is the first three columns of A, $\mathsf{RREF}(B)$ is the first three columns of $\mathsf{RREF}(A)$, so the answer is

$$\boxed{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}.$$

(c) What is the reduced echelon form of the matrix $C = \begin{bmatrix} v_1 & v_3 & v_5 \end{bmatrix}$?

Solution:

Since 1, 3 and 5 are pivot columns, v_1, v_3, v_5 are linearly independent, so C must have a pivot in every column and RREF(C) must be

$$\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}.$$

(d) What is the reduced echelon form of the matrix $D = \begin{bmatrix} v_3 & v_4 & v_5 \end{bmatrix}$?

Solution:

D is the last three columns of A, so D is **row equivalent** to the last three columns of RREF(A). This submatrix is not yet in reduced echelon form, but a few row operations gets it there:

$$\begin{bmatrix} 0 & 3 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So the answer is again

$$\boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}.$$

(e) Find the general solution to the linear system

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 + 4x_3 + 2x_4 = 0 \\ 2x_1 - 4x_3 + x_4 = 0. \end{cases}$$

Solution:

The augmented matrix of this system is

$$A = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 & 0 \\ 2 & 0 & -4 & 1 & 0 \end{array}\right].$$

We compute its reduced echelon form by the sequence of row operations

$$A = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 2 & 0 \\ 2 & 0 & -4 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & -2 & -6 & -1 & -2 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 1 & -1 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 5 \\ 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 1 & -4 \end{array}\right] = \mathsf{RREF}(A).$$

We see that 1, 2, and 4 are pivot columns, so the x_1, x_2, and x_4 are basic variables, x_3 is a free variable, and there are infinitely many solutions. The linear system with augmented matrix $\mathsf{RREF}(A)$ has the same solutions as our starting system, and is given by

$$\begin{cases} x_1 - 2x_3 = 2 \\ x_2 + 3x_3 = 3 \\ x_4 = -4 \end{cases}$$

so the general solution is $\boxed{(x_1, x_2, x_3, x_4) = (2 + 2a, 3 - 3a, a, -4) \text{ where } a \in \mathbb{R} \text{ is any real number}}$.

Problem 2. (4 + 4 + 4 + 4 + 4 = 20 points)

Recall that the standard matrix of a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ is the matrix A such that $f(x) = Ax$ for all $x \in \mathbb{R}^n$. This problem has five parts.

(a) Find the standard matrix of the linear function $f : \mathbb{R}^4 \to \mathbb{R}^2$ defined by

$$f\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}\right) = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Solution:

The standard matrix of f is

$$\left[\begin{array}{cccc} f\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right) & f\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right) & f\left(\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) & f\left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right) \end{array}\right]$$

which we compute to be $\boxed{\begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 0 & 2 & 3 \end{bmatrix}}$.

(b) Find the standard matrix of the linear function $f : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$f\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \det \begin{bmatrix} 1 & v_1 & 2 \\ 5 & v_2 & 4 \\ 2 & v_3 & 3 \end{bmatrix}.$$

Solution:

Since $f(v) = (3v_2 - 4v_3) - v_1(15 - 8) + 2(5v_3 - 2v_2) = -7v_1 - v_2 + 6v_3$ the standard matrix is $\boxed{\begin{bmatrix} -7 & -1 & 6 \end{bmatrix}}$.

(c) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}^2$ are both linear functions.

Recall that $f \circ g$ is the function defined by $(f \circ g)(x) = f(g(x))$.

If f has standard matrix $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and $f \circ g$ has standard matrix $\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$ then what is the standard matrix of g?

Solution:

Let A and B be the standard matrices of f and g. Then $A = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ and $AB = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$ so $B = A^{-1}(AB) = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} -9 & 10 \\ 6 & -6 \end{bmatrix}}$.

(d) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear function with standard matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \end{bmatrix}.$$

Determine m, n, and whether or not f is one-to-one or onto.

$m = 2$	$n = 3$

Is f one-to-one? No

Justify your answer:

A linear function $\mathbb{R}^n \to \mathbb{R}^m$ with $n > m$ is never one-to-one.

Is f onto? Yes

Justify your answer:

f is onto since A has a pivot in every row, as $\mathsf{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(e) Suppose $a, b, c, d \in \mathbb{R}$ are real numbers with $ad - bc \neq 0$. There is a unique linear function $f : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$f\left(\begin{bmatrix} a \\ c \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad f\left(\begin{bmatrix} b \\ d \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

What is the standard matrix of f?

Solution:

Let A be the standard matrix of f. If g is the linear function $\mathbb{R}^2 \to \mathbb{R}^2$ with

$$g\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} a \\ c \end{bmatrix} \quad \text{and} \quad g\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} b \\ d \end{bmatrix}$$

then g has standard matrix $B = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$. We have $AB = I$ since

$$AB\begin{bmatrix} 1 \\ 0 \end{bmatrix} = f\left(g\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\right) = f\left(\begin{bmatrix} b \\ d \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and

$$AB\begin{bmatrix} 0 \\ 1 \end{bmatrix} = f\left(g\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)\right) = f\left(\begin{bmatrix} a \\ c \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

so $A = B^{-1} = \boxed{\frac{1}{bc - ad}\begin{bmatrix} c & -a \\ -d & b \end{bmatrix}}$.

Problem 3. (4 + 4 + 4 + 8 = 20 points)

Suppose A is a (not necessarily square) $m \times n$ matrix. Note that $A^T A$ is a square $n \times n$ matrix. Here are some possible properties that A could have:

(1) A is square with $m = n$.
(2) every row of A has a pivot position.
(3) every column of A has a pivot position.
(4) the equation $Ax = b$ has a solution for each $b \in \mathbb{R}^m$.
(5) the equation $Ax = b$ has a unique solution for each $b \in \mathbb{R}^m$.

(6) the equation $Ax = b$ has a unique solution for some $b \in \mathbb{R}^m$.
(7) the equation $Ax = 0$ has infinitely many solutions.
(8) the equation $Ax = 0$ has exactly one solution.
(9) the equation $Ax = 0$ does not have a solution.
(10) $\mathrm{Col}(A) = \mathbb{R}^m$.

(11) $\mathrm{Nul}(A) = \mathbb{R}^n$.
(12) $\mathrm{Col}(A) = \{0\}$.
(13) $\mathrm{Nul}(A) = \{0\}$.
(14) the columns of A are linearly independent.
(15) the span of the columns of A is $\mathbb{R}^m$.

(16) the reduced echelon form of A is an identity matrix.
(17) AB is an identity matrix for some $n \times m$ matrix B.
(18) $\mathrm{rank}(A) = \min\{m, n\}$.
(19) $\det(A^T A) \neq 0$.
(20) $(A^T A)^k$ is an identity matrix for some integer $k \geq 1$.

(a) Circle all of the properties that A must have **if A is invertible**.

$\boxed{1}$	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$
$\boxed{6}$	7	$\boxed{8}$	9	$\boxed{10}$
11	12	$\boxed{13}$	$\boxed{14}$	$\boxed{15}$
$\boxed{16}$	$\boxed{17}$	$\boxed{18}$	$\boxed{19}$	20

(19) holds because if A is invertible then A is square with $\det(A) \neq 0$ so $\det(A^T A) = \det(A^T)\det(A) = \det(A)^2 \neq 0$.

(20) does not always hold as A could be the 1×1 matrix $\begin{bmatrix} 2 \end{bmatrix}$, for example.

(b) Circle all of the properties that **imply by themselves that A is invertible**.

1	2	3	4	$\boxed{5}$
6	7	8	9	10
11	12	13	14	15
$\boxed{16}$	17	18	19	20

The key thing to remember here is that A is not assumed to be square.

(c) Circle all of the properties that **imply by themselves that A is invertible if we also assume that A is a square matrix with** $m = n$.

	$\boxed{2}$	$\boxed{3}$	$\boxed{4}$	$\boxed{5}$
$\boxed{6}$	7	$\boxed{8}$	9	$\boxed{10}$
11	12	$\boxed{13}$	$\boxed{14}$	$\boxed{15}$
$\boxed{16}$	$\boxed{17}$	$\boxed{18}$	$\boxed{19}$	$\boxed{20}$

(19) is sufficient since if A is square then $\det(A)^2 = \det(A^T A)$ so $\det(A)$ is nonzero (and A is invertible) whenever $\det(A^T A)$ is nonzero.

(20) is sufficient since if $(A^T A)^k$ is an identity matrix for $k \geq 1$ then $(A^T A)^{k-1} A^T A = (A^T A)^k = I$ so A is invertible with $A^{-1} = (A^T A)^{k-1} A^T$.

(d) Compute the inverse of the matrix

$$A = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 4 & 4 & 4 & 4 \end{bmatrix}.$$

Solution:

We row reduce

$$[\ A \mid I\ ] = \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 4 & 4 & 4 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 4 & 4 & \mathbf{0} & -\mathbf{1} & 0 & 0 & 1 \end{array}\right] \text{(subtract row 1 from row 4)}$$

$$\to \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ \mathbf{0} & \mathbf{0} & 4 & 0 & -1 & 0 & -\mathbf{2} & 1 \end{array}\right] \text{(subtract twice row 3 from row 4)}$$

$$\to \left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \mathbf{0} & 2 & 0 & 0 & 0 & -\mathbf{2} & 1 & 0 \\ 0 & 0 & 4 & 0 & -1 & 0 & -2 & 1 \end{array}\right] \text{(subtract twice row 2 from row 3)}$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 4 & 0 & -1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 \end{array}\right] \text{(rearrange rows)}$$

$$\to \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & -1/4 & 0 & -1/2 & 1/4 \\ 0 & 0 & 0 & 1 & 1/4 & 0 & 0 & 0 \end{array}\right] = [\ I \mid A^{-1}\ ]$$

to get the answer

$$\boxed{A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 1/2 & 0 \\ -1/4 & 0 & -1/2 & 1/4 \\ 1/4 & 0 & 0 & 0 \end{bmatrix}}.$$

Problem 4. (2 + 6 + 6 + 6 = 20 points)

The matrix

$$A = \begin{bmatrix} 2 & 4 & 2 & 0 & 2 \\ 1 & 2 & 0 & 1 & 3 \\ 2 & 4 & 2 & 0 & 2 \\ 1 & 2 & 2 & -1 & -1 \end{bmatrix}$$

has reduced echelon form

$$\mathsf{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This question has four parts.

There was a typo in this problem: we should have $\mathsf{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

If we ignore this and assume $\mathsf{RREF}(A)$ is as given, then the solution is as follows:

(a) Find a basis for the column space of A.

Solution:

One basis for the column space consists of the pivot columns of A, namely columns 1 and 3:

$$\boxed{\left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \end{bmatrix} \right\}}.$$

(b) Find a basis for the null space of A.

Solution:

The linear system $\mathsf{RREF}(A)x = 0$, omitting trivial equations $0 = 0$, can be written as

$$\begin{cases} x_1 = -2x_2 - x_4 - 3x_5 \\ x_3 = x_4 - 2x_5. \end{cases}$$

This tells us that if $Ax = 0$, which holds if and only if $\mathsf{RREF}(A)x = 0$, then

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_4 - 3x_5 \\ x_2 \\ x_4 - 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

so a basis for $\mathrm{Nul}(A)$ is

$$\boxed{\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}}.$$

(c) Find all pairs of real numbers (x, y) such that $\begin{bmatrix} 2 \\ x \\ 2 \\ y \end{bmatrix}$ is in $\mathrm{Col}(A)$.

Solution:

In view of part (a), the given vector is in the column space if and only if there are coefficients $c_1, c_2 \in \mathbb{R}$ such that

$$c_1 \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2(c_1 + c_2) \\ c_1 \\ 2(c_1 + c_2) \\ c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ x \\ 2 \\ y \end{bmatrix}.$$

For this to hold we must have $c_1 + c_2 = 1$ and $c_1 = x$ and $c_1 + 2c_2 = y$. This means that $y = c_1 + 2c_2 = 2(c_1 + c_2) - c_1 = 2 - x$. So if the vector is in the column space then we must have $y = 2 - x$.

Conversely, as long as $y = 2 - x$, the desired coefficients are given by $c_1 = x$ and $c_2 = 1 - x$, so the vector is in the column space. So the vector is in column space precisely when $\boxed{(x, y) \text{ is any pair of real numbers with } y = 2 - x}$.

(d) Find all pairs of real numbers (x, y) such that $\begin{bmatrix} x \\ 0 \\ y \\ 1 \\ 1 \end{bmatrix}$ is in $\mathrm{Nul}(A)$.

Solution:

This vector is in $\mathrm{Nul}(A)$ if and only if it is in $\mathrm{Nul}(\mathsf{RREF}(A))$. But

$$\mathsf{RREF}(A) \begin{bmatrix} x \\ 0 \\ y \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \\ y \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} x + 4 \\ y + 1 \\ 0 \\ 0 \end{bmatrix}$$

is zero precisely when $\boxed{(x, y) = (-4, -1)}$.

If we instead use the correct value of $\mathsf{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

then the solution is almost the same:

(a) (same solution)

(b) $\boxed{\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}}$.

(c) (same solution)

(d) $\boxed{(x, y) = (-4, 3)}$.

When grading, we also gave full points if you got these answers.