MIDTERM – MATH 2121, FALL 2021.

Name:	
Student ID:	
Email:	

Section: L1 (online) L2 (in person)

Tutorial: T1A T1B T1C T2A T2B T2C

Problem #	Points Possible	Score
1	10	
2	15	
3	10	
4	5	
5	15	
6	10	
7	15	
Total	80	

You have **120 minutes** to complete this exam.

No books, notes, or electronic devices can be used on the test.

Draw a box around your answers or write your answers in the boxes provided.

Partial credit can be given on some problems if you show your work. Good luck!

Problem 1. (10 points)

Suppose A is a 2×3 matrix whose columns span $\mathbb{R}^2$.

(a) Describe all matrices that could occur as the reduced echelon form of A. Be as specific as possible.

(b) Suppose further that $A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for some $a, b, c \in \mathbb{R}$ with $c \neq 0$.

Describe all matrices that could occur as the reduced echelon form of A.

Solution:

Problem 2. (15 points)

Suppose a and b are real numbers. Consider the lines

$$L_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = ax \right\} \quad \text{and} \quad L_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : y = bx \right\}.$$

(a) When is it impossible to express the vector $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ as a sum of two vectors, one on the line L_1 and one on the line L_2?

(b) When is there more than one way of expressing the vector $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ as a sum of two vectors, one on the line L_1 and one on the line L_2?

(c) When is there exactly one way of writing

$$\begin{bmatrix} 5 \\ 6 \end{bmatrix} = v + w$$

with $v \in L_1$ and $w \in L_2$? Find a formula for v and w in this case.

Your answers to each part should be in terms of a and b.

Solution:

Problem 3. (10 points)

(a) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that rotates a vector counterclockwise by 45 degrees and then doubles its length. Find the standard matrix of T, that is, the matrix A such that $T(v) = Av$ for all $v \in \mathbb{R}^2$.

(b) Let M be a 2×2 rotation matrix not equal to the identity matrix.

Suppose $M^{-1} = M^5$ and $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

How many different vectors could be in the set

$$S = \{v,\ Mv,\ M^2v,\ M^3v,\ M^4v,\ M^5v\}?$$

For each possibility, draw a picture representing the vectors in S and compute the sum $v + Mv + M^2v + M^3v + M^4v + M^5v$.

You do not need to write down numeric expressions for the vectors in S.

Solution:

Problem 4. (5 points)

Find the value(s) of $h \in \mathbb{R}$ for which the following vectors are linearly dependent:

$$\begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} -6 \\ 8 \\ 7 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ -2 \\ h \end{bmatrix}.$$

Solution:

Problem 5. (15 points)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.
Suppose H is a k-dimensional subspace of $\mathbb{R}^n$. Define the set

$$T(H) = \{T(v) : v \in H\}.$$

(a) Explain why $T(H)$ is a subspace of $\mathbb{R}^m$.

(b) If T is onto, then what are the possibilities for $\dim T(H)$?
Justify your answer, which should be in terms of k, m, and n.

(c) If T is one-to-one, then what are the possibilities for $\dim T(H)$?
Justify your answer, which should be in terms of k, m, and n.

Solution:

Problem 6. (10 points) Suppose $A = \begin{bmatrix} u & v & w & x & y & z \end{bmatrix}$ is a 4×6 matrix with columns $u, v, w, x, y, z \in \mathbb{R}^4$. The reduced echelon form of A is

$$\mathsf{RREF}(A) = \begin{bmatrix} 0 & 1 & 0 & 3 & 0 & -2 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find a basis for the null space of A.
(b) Find a basis for the column space of A.

Problem 7. (15 points)

Let $n \geq 2$ be a positive integer. Suppose A is the $n \times n$ matrix with 0's on the main diagonal and 1's everywhere else. For example, if $n = 4$ then we would have

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Let I be the $n \times n$ identity matrix.

(a) Find numbers b and c such that $A^2 = bI + cA$. (These will depend on n.)
(b) Compute a formula for the inverse of A. Be as specific as possible.
(c) Compute a formula for $\det(A)$.

Solution: