MORE FINAL REVIEW PROBLEMS – MATH 2121

Below are some more exercises to help you review for our final examination.

Exercise 1. Find a general formula for all solutions to the linear system

$$\begin{aligned} x_1 + 5x_3 &= 4 \\ 2x_1 + x_2 + 6x_3 &= 4 \\ 3x_1 + 4x_2 - x_3 &= -4 \end{aligned}$$

Solution:

Exercise 2. Express the vector $b = \begin{bmatrix} 2 \\ 13 \\ 6 \end{bmatrix}$ as a linear combination of the vectors

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad v = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \qquad w = \begin{bmatrix} 5 \\ 6 \\ 0 \end{bmatrix}.$$

Solution:

Exercise 3. Show that the vector $b = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$ is not in the span of the vectors

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad v = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \qquad w = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}.$$

Solution:

Exercise 4. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation with

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad T\left(\begin{bmatrix} 5 \\ 6 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Find the standard matrix A for T, which satisfies $T(v) = Av$ for all $v \in \mathbb{R}^3$.

Solution:

Exercise 5. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a function

(a) Write down what it means for T to be *linear*.

(b) Write down what it means for T to be *one-to-one*.
Explain how to determine if T is one-to-one when T is linear.

(c) Write down what it means for T to be *onto*.
Explain how to determine if T is onto when T is linear.

Solution:

Exercise 6. Compute the matrix products

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.$$

Solution:

Exercise 7. Find the inverse of $A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 2 & 4 \\ 0 & 2 & 3 \end{bmatrix}$.

Solution:

Exercise 8. Write in your own words definitions to the following vocabulary:

(1) A *linear combination* of some vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$.
(2) The *span* of some vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$.
(3) A *linearly independent* set of vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$.
(4) A *linearly dependent* set of vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$.
(5) A *subspace* of $\mathbb{R}^n$.
(6) A *basis* of a subspace of $\mathbb{R}^n$.
(7) The *dimension* of a subspace of $\mathbb{R}^n$.
(8) The *column space* of a matrix A.
(9) The *null space* of a matrix A
(10) The *rank* of a matrix A.

Solution:

Exercise 9. Find bases for $\mathrm{Col}A$ and $\mathrm{Nul}A$ when $A = \begin{bmatrix} 6 & 3 & 6 & 9 \\ 4 & 2 & 4 & 6 \\ 6 & 3 & 5 & 9 \end{bmatrix}$.

Solution:

Exercise 10. Consider the matrix

$$A = \begin{bmatrix} 6 & 1 & 1 \\ 4 & -2 & 5 \\ 2 & 8 & 7 \end{bmatrix}.$$

(a) Compute $\det A$ using the formula

$$\det A = \sum_{X \in S_3} \mathsf{prod}(X, A)(-1)^{\mathsf{inv}(X)}.$$

(b) Compute $\det A$ using the row reduction algorithm discussed in Lecture 12.

(c) Compute $\det A$ using the formula

$$\det A = a_{11} \det A^{(1,1)} - a_{21} \det A^{(2,1)} + a_{31} A^{(3,1)}$$

discussed at the end of Lecture 12.

(d) Without doing any (significant) calculation, compute

$$\det A^{-1}, \qquad \det A^T, \qquad \det B, \qquad \text{and} \qquad \det C$$

for the matrices

$$B = \begin{bmatrix} 1 & 1 & 6 \\ 5 & -2 & 4 \\ 7 & 8 & 2 \end{bmatrix} \qquad \text{and} \qquad C = \begin{bmatrix} 12 & 1 & 2 \\ 8 & -2 & 3 \\ 4 & 8 & 15 \end{bmatrix}.$$

Solution:

Exercise 11. Find all (possibly complex) eigenvalues for the matrices

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Are these matrices similar?

Solution:

Exercise 12. Diagonalize the matrix

$$A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix}.$$

In other words, find an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}$$

Use this to compute exact formulas for the functions defined by

$$\begin{bmatrix} a(n) & b(n) \\ c(n) & d(n) \end{bmatrix} = A^n$$

for positive integers $n = 1, 2, 3, \dots$.

Finally, calculate the limit $\lim_{n\to\infty} A^n$.

Solution:

Exercise 13. Find the rank and eigenvalues of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Solution:

Exercise 14. Find the eigenvalues and determinants of

$$B = A - I = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C = I - A = \begin{bmatrix} 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$

Solution:

Exercise 15. Consider the vector space

$$V = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$$

of polynomials in one variable x with degree at most three.

(a) Define $T : V \to V$ to be the function with $T(f(x)) = f(x+1)$ for $f \in V$, so

$$T(3x) = 3x + 3 \quad \text{and} \quad T(x^2) = x^2 + 2x + 1,$$

for example. Explain why this function is linear.

(b) Let $A : \mathbb{R}^3 \to V$ and $B : V \to \mathbb{R}^3$ be the linear functions with

$$A(e_i) = x^{i-1} \quad \text{and} \quad B(x^{i-1}) = e_i \quad \text{for } i \in \{1, 2, 3\}$$

where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The composition $F = B \circ T \circ A$ is a linear function $\mathbb{R}^3 \to \mathbb{R}^3$.
Determine the standard matrix of F.

(c) Using part (b), find all eigenvalues for T and for each eigenvalue find a corresponding eigenvector.

In this context, an eigenvector for T with eigenvalue λ is a nonzero polynomial $f(x) = ax^2 + bx + c \in V$ such that

$$T(f(x)) = f(x+1) = \lambda f(x)$$

which is equivalent to

$$a(x+1)^2 + b(x+1) + c = (\lambda a)x^2 + (\lambda b)x + (\lambda c).$$

Solution:

Exercise 16.

(a) Draw a picture representing a subspace V, a vector b, and the orthogonal projection $\mathrm{proj}_V(b)$ of b onto V (say, in $\mathbb{R}^3$).

(b) Suppose A is an $m \times n$ matrix and $b \in \mathbb{R}^m$.
Assume the linear system $Ax = b$ is inconsistent.

Draw a picture representing $\mathrm{Col}A$ and b and $\mathrm{proj}_{\mathrm{Col}A}(b)$.

Use this picture to explain why the equation $Ax = \mathrm{proj}_{\mathrm{Col}A}(b)$ always has a solution and why a solution to this equation minimizes $\|Ax - b\|$.

(This shows that the exact solutions to $Ax = \mathrm{proj}_{\mathrm{Col}A}(b)$ are the least-squares solutions to $Ax = b$. We showed in class that the exact solutions to $Ax = \mathrm{proj}_{\mathrm{Col}A}(b)$ are the same as the exact solutions to $A^T Ax = A^T b$.)

Solution:

Exercise 17. There are three parts to this problem.

(a) Find an orthogonal basis for the column space of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 2 \\ 2 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

(b) Find the orthogonal projection of the vector $v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto $\mathrm{Col}(A)$.

(c) Finally, find a basis for $\mathrm{Col}(A)^\perp$.

Solution:

Exercise 18. Suppose a function $f : \mathbb{R} \to \mathbb{R}$ has the following values:

x	$f(x)$
0	0
1	6
2	5
3	10
4	7

Find $a, b, c, d \in \mathbb{R}$ such that the cubic equation

$$y = ax^3 + bx^2 + cx + d$$

best approximates $f(x)$ in the sense of least-squares.

Solution:

Exercise 19. Consider the symmetric matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}.$$

Find an orthogonal matrix U and a diagonal matrix D such that

$$A = UDU^T.$$

Solution:

Exercise 20. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

Solution: