Instructions: Choose **4 problems** and write down detailed solutions, showing all necessary work. You can earn up to **4 extra credit points** by correctly solving additional problems.[1]

Some of the problems are more challenging than others, and there is no need to solve all of them. Problems that would not make reasonable exam questions (either because of difficulty, being open-ended, or requiring external resources) are marked with a star. These problems may still offer useful practice with the core concepts in the course.

You are free to discuss problems with other students and to consult whatever resources you want, but you must write up your own solutions. If your solutions appear to be copied from somewhere else, you will automatically receive zero credit. **Please handwrite your answers and show all steps in your calculations**, as you would on an exam.

To get full credit for the offline homework, you just need to make a good-faith attempt on the required problems. The bar for receiving extra credit points is higher.

1. (a) For each $i = 1, 2, 3$, the map $T_i : \mathbb{R}^3 \to \mathbb{R}^2$ given by the formula

$$T_i(v) = (\text{the vector } v \text{ with row } i \text{ deleted})$$

   is a linear transformation. What are the standard matrices of T_1, T_2, and T_3?

   (b) Can you find two linearly independent vectors $v, w \in \mathbb{R}^3$ such that for each $i = 1, 2, 3$, if we delete row i from both vectors, we get linearly independent vectors in $\mathbb{R}^2$?

   In other words, such that the set $\{T_1(v), T_1(w)\}$ is linearly independent, $\{T_2(v), T_2(w)\}$ is linearly independent, and $\{T_3(v), T_3(w)\}$ is linearly independent. Justify your answer.

   (c) Can you find two linearly independent vectors $v, w \in \mathbb{R}^3$ such that for each $i = 1, 2, 3$, if we delete row i from both vectors, we get linearly dependent vectors in $\mathbb{R}^2$?

   In other words, such that the set $\{T_1(v), T_1(w)\}$ is linearly dependent, $\{T_2(v), T_2(w)\}$ is linearly dependent, and $\{T_3(v), T_3(w)\}$ is linearly dependent. Justify your answer.

*2. Suppose A is an $m \times n$ matrix and B is an $m \times q$ matrix.

   The *rank* of a matrix M, denote rank(M), is its number of pivot columns.

   (a) What relationship, if any, exists between $\mathsf{RREF}(A)$ and $\mathsf{RREF}(\begin{bmatrix} A & B \end{bmatrix})$ in general?

   What relationship, if any, exists between $\text{rank}(A)$ and $\text{rank}(\begin{bmatrix} A & B \end{bmatrix})$ in general?

   (b) What relationship, if any, exists between $\mathsf{RREF}(B)$ and $\mathsf{RREF}(\begin{bmatrix} A & B \end{bmatrix})$ in general?

   What relationship, if any, exists between $\text{rank}(B)$ and $\text{rank}(\begin{bmatrix} A & B \end{bmatrix})$ in general?

   (c) What relationship, if any, exists between $\mathsf{RREF}(\begin{bmatrix} A & B \end{bmatrix})$ and $\mathsf{RREF}(\begin{bmatrix} B & A \end{bmatrix})$ in general?

   What relationship, if any, exists between $\text{rank}(\begin{bmatrix} A & B \end{bmatrix})$ and $\text{rank}(\begin{bmatrix} B & A \end{bmatrix})$ in general?

   For the first question in each part, the possible answers are

   "the two RREFs are equal" or
   "the smaller RREF is a specific submatrix of the larger RREF" or
   "no general relationship".

   For the second question in each part, the possible answers are: $\leq$, $\geq$, $=$, or no general relationship.

   Just pick from among these options, and provide some explanation for your choices.

*3. Suppose A is an $m \times n$ matrix and C is a $q \times n$ matrix.

[1] There will be ∼10 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼40 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

(a) What relationship, if any, exists between $\mathsf{RREF}(A)$ and $\mathsf{RREF}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ in general?

What relationship, if any, exists between $\text{rank}(A)$ and $\text{rank}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ in general?

(b) What relationship, if any, exists between $\mathsf{RREF}(C)$ and $\mathsf{RREF}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ in general?

What relationship, if any, exists between $\text{rank}(C)$ and $\text{rank}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ in general?

(c) What relationship, if any, exists between $\mathsf{RREF}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ and $\mathsf{RREF}\left(\begin{bmatrix} C \\ A \end{bmatrix}\right)$ in general?

What relationship, if any, exists between $\text{rank}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ and $\text{rank}\left(\begin{bmatrix} C \\ A \end{bmatrix}\right)$ in general?

Use the same set of possible answers as in the previous question.

4. Suppose $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ are vectors and $V = \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_k\}$.

   (a) If we add another vector v_{k+1} to this list, will it still have the same span?

   What can you say about when $V = \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_k, v_{k+1}\}$?

   (b) If we delete one of the vectors from the list, say v_k, will it still have the same span?

   What can you say about when $V = \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_{k-1}\}$?

   Explain and justify your answers to both parts.

5. Let $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$.

   The span $\mathbb{R}\text{-span}\{v, w\} = \{\alpha v + \beta w : \alpha, \beta \in \mathbb{R}\}$ is the smallest set of vectors in $\mathbb{R}^2$ containing v and w that is **closed under scalar multiplication and vector addition/subtraction**.

   (a) Explain why $\mathbb{R}\text{-span}\{v, w\} = \mathbb{R}^2$.

   (b) Draw a picture of the smallest set $\mathcal{S}$ of vectors in $\mathbb{R}^2$ that contains v and w and is closed under just scalar multiplication. This means that $v \in \mathcal{S}$, $w \in \mathcal{S}$, if $u \in \mathcal{S}$ then $cu \in \mathcal{S}$ for all $c \in \mathbb{R}$, and $\mathcal{S}$ is as small as possible with these properties.

   (c) Draw a picture of the smallest set $\mathcal{T}$ of vectors in $\mathbb{R}^2$ that contains v and w and is closed under just vector addition and subtraction. This means that $v \in \mathcal{T}$, $w \in \mathcal{T}$, if $x, y \in \mathcal{T}$ then $x + y \in \mathcal{T}$ and $x - y \in \mathcal{T}$, and $\mathcal{T}$ is as small as possible with these properties.

   (d) Draw pictures of $\frac{1}{2}\mathcal{S} = \{\frac{1}{2}u : u \in \mathcal{S}\}$ and $\frac{1}{2}\mathcal{T} = \{\frac{1}{2}u : u \in \mathcal{T}\}$.

   In general, how is $\mathcal{S}$ related to $\frac{1}{n}\mathcal{S}$ and how is $\mathcal{T}$ related to $\frac{1}{n}\mathcal{T}$ when n is a positive integer?

*6. Suppose $v_1, v_2, \dots, v_k$ and $w_1, w_2, \dots, w_l$ are two lists of vectors in $\mathbb{R}^n$.

   Describe an algorithm to determine whether or not $\mathbb{R}\text{-span}\{v_1, v_2, \dots, v_k\} = \mathbb{R}\text{-span}\{w_1, w_2, \dots, w_l\}$.

7. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation.

   If $T\left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 4 \\ 4 \\ 7 \end{bmatrix}\right)$ then what is $T\left(\begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix}\right)$?

   Justify your answer.

8. Suppose $m \leq n$ are positive integers.

   Let X be a finite set with m elements and let Y be a finite set with n elements.

Let $\mathbb{Z}$ be the infinite set of all integers.

(a) Explain why any one-to-one map $f : Y \to X$ is also onto.

Similarly, explain why any one-to-one linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ is also onto.

(b) Explain why any onto map $f : X \to Y$ is also one-to-one.

Similarly, explain why any onto linear map $f : \mathbb{R}^m \to \mathbb{R}^n$ is also one-to-one.

(c) These properties do not hold for arbitrary maps between infinite sets.

To show this, give an example of a map $f : \mathbb{Z} \to \mathbb{Z}$ that is one-to-one but not onto.

Then give an example of a map $g : \mathbb{Z} \to \mathbb{Z}$ that is onto but not one-to-one.

(Suggestion: for parts (a) and (b) use the characterization of onto and one-to-one linear maps in terms of the reduced echelon form of the standard matrix.)

9. Consider the set of vectors $\mathcal{Q}$ in $\mathbb{R}^2$ whose endpoints are in the quadrilateral in the xy-plane with vertices $(0, 0)$, $(-20, 23)$, $(21, 21)$, and $(1, 44)$.

If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with $\left\{ T\left(\left[\begin{array}{c} a \\ b \end{array}\right]\right) : 0 \leq a \leq 1 \text{ and } 0 \leq b \leq 1 \right\} = \mathcal{Q}$, then what are the possibilities for the standard matrix A of T? Justify your answer.

*10. The unit cube in $\mathbb{R}^n$ is the set $\mathcal{C} = \{v \in \mathbb{R}^n : 0 \leq v_i \leq 1 \text{ for all } i = 1, 2, \ldots, n\}$.

Find all $n \times n$ matrices A such that $A\mathcal{C} = \mathcal{C}$. Be sure to justify your answer.

Here $A\mathcal{C}$ is defined to be the set $\{Av : v \in \mathcal{C}\}$.