Instructions: Choose **4 problems** and write down detailed solutions, showing all necessary work. You can earn up to **4 extra credit points** by correctly solving additional problems.[1]

Some of the problems are more challenging than others, and there is no need to solve all of them. Problems that would not make reasonable exam questions (either because of difficulty, being open-ended, or requiring external resources) are marked with a star. These problems may still offer useful practice with the core concepts in the course.

You are free to discuss problems with other students and to consult whatever resources you want, but you must write up your own solutions. If your solutions appear to be copied from somewhere else, you will automatically receive zero credit. **Please handwrite your answers and show all steps in your calculations**, as you would on an exam.

To get full credit for the offline homework, you just need to make a good-faith attempt on the required problems. The bar for receiving extra credit points is higher.

Show all steps and provide justification for all answers.

1. Define the *two-sided reduced echelon form* of a matrix A to be

$$\mathsf{TREF}(A) := \mathsf{RREF}(\mathsf{RREF}(A)^T)^T.$$

Here M^T denotes the transpose of a matrix M.

As a warmup, calculate $\mathsf{TREF}(A)$ if

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}.$$

Describe all the possibilities for $\mathsf{TREF}(A)$ when A is a general $m \times n$ matrix.

**2. Does it always hold that $\mathsf{TREF}(A^T) = \mathsf{TREF}(A)^T$ for any matrix A?

Prove this property or find a counterexample.

This problem is double starred because I don't think there is any simple solution using only what we have seen in the course so far. I am leaving this in as a challenge, but only attempt if interested!

3. Find a general formula for A^n when $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and n is any positive integer.

(Hint: write $A = I + B$ where I the identity matrix. Then find a general formula for B^n first.)

4. Suppose $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R}$. When is

$$A = \begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{bmatrix}$$

invertible? Find a formula for A^{-1} when A is invertible.

[1] There will be ∼10 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼40 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

5. Suppose $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R}$. When is

$$A = \begin{bmatrix} 0 & 0 & a_1 & b_1 \\ 0 & 0 & c_1 & d_1 \\ a_2 & b_2 & 0 & 0 \\ c_2 & d_2 & 0 & 0 \end{bmatrix}$$

invertible? Find a formula for A^{-1} when A is invertible.

6. Suppose $a, b \in \mathbb{R}$ and $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $D = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}$.

   Find an invertible matrix P such that $AP = PD$.

   Then find a general formula for A^n when n is any positive integer.

   (Hint: first express A, A^2, and A^3 in terms of P and D.)

*7. Suppose $a, b, c, d \in \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ to be the function with formula

$$f(x) = ax^3 + bx^2 + cx + d.$$

   Important note: we are defining f to have domain $\mathbb{R}$ and also codomain $\mathbb{R}$.

   What are the possibilities for (a, b, c, d) if f is an invertible function $\mathbb{R} \to \mathbb{R}$?

   (Hint: separately consider the cases when $a \neq 0$, $a = 0 \neq b$, $a = b = 0 \neq c$, etc., and use calculus.)

8. Suppose A is an $n \times n$ matrix and I is the $n \times n$ identity matrix.

   Choose positive real numbers b and c.

   (a) Assume $A^2 = bI + cA$. Find a formula for A^{-1}.

   (b) Suppose A is the $n \times n$ matrix with b on the main diagonal and c everywhere else.

   For example, if $n = 4$ then we would have

$$A = \begin{bmatrix} b & c & c & c \\ c & b & c & c \\ c & c & b & c \\ c & c & c & b \end{bmatrix}.$$

   When is A invertible? Compute A^{-1} when A is invertible.

   (Your answer should be for general $b, c > 0$ and general n, not just for $n = 4$.)

9. Suppose A and B are $n \times n$ matrices such that $v^T A w = v^T B w \in \mathbb{R}$ for all vectors $v, w \in \mathbb{R}^n$.

   Does it always hold that A and B are equal? Prove that $A = B$ or find a counterexample.

10. The *commutator* of two invertible matrices A and B is $[A, B] := ABA^{-1}B^{-1}$.

   Compute a formula for $[A, B]$ when $A = \begin{bmatrix} 1 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & v & w & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   Here $a, b, c, d, e, v, w, x, y, z$ are arbitrary real numbers.

11. Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Find all 3×2 matrices B such that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

12. Suppose A is an $m \times n$ matrix where $m \neq n$ and I is the $m \times m$ identity matrix. Describe an algorithm to determine if there exists a $n \times m$ matrix B with $AB = I$, and to find one such matrix B when one does exist. Explain why your algorithm works.