Instructions: *Complete the following exercises. Solutions will be graded on clarity as well as correctness. Feel free to discuss the problems with other students, but be sure to acknowledge your collaborators in your solutions, and to write up your final solutions by yourself.*

Due on **Thursday, March 17**.

Except when mentioned otherwise, all Lie algebras and vector spaces below are defined over an algebraically closed field $\mathbb{F}$ of characteristic zero.

1. Let A be a finite-dimensional $\mathbb{F}$-algebra. Recall that $\operatorname{Der} A$ is defined to be the set of linear maps $\delta : A \to A$ satisfying $\delta([X, Y]) = [X, \delta(Y)] + [\delta(X), Y]$ for $X, Y \in A$, where $[X, Y] = XY - YX$.

   Each $X \in \operatorname{Der} A$ is a linear map $A \to A$ so has a unique Jordan decomposition $X = X_s + X_n$.

   Here X_s and X_n are also linear maps $A \to A$ satisfying some properties.

   Prove that actually $X_s \in \operatorname{Der} A$ and $X_n \in \operatorname{Der} A$.

   (Fill in the details to the proof of Lemma 4.2B in textbook.)

   Conclude that if L is a semisimple Lie algebra of finite dimension then for each $X \in L$ there are unique elements $X_s, X_n \in L$ with $\operatorname{ad}(X_s) = (\operatorname{ad} X)_s$ and $\operatorname{ad}(X_n) = (\operatorname{ad} X)_n$.

2. Let m be a nonnegative integer and let $V(m)$ be a vector space with basis $v_0, v_1, v_2, \ldots, v_m$. Define $Hv_i = (m - 2i)v_i$ and $Yv_i = (i + 1)v_{i+1}$ and $Xv_i = (m - i + 1)v_{i-1}$ where $v_{-1} = v_{m+1} = 0$. Show that these formulas extend to a module structure for the Lie algebra $\mathfrak{sl}_2(\mathbb{F})$ where

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and} \quad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

   To check this, verify that the matrices describing the action of H, Y, and X on $V(m)$ satisfy the same Lie bracket equations as H, Y, and X do.

3. $M = \mathfrak{sl}_3(\mathbb{F})$ contains a copy of $\mathfrak{sl}_2(\mathbb{F})$ in its upper left 2×2 position. We can view M as an $\mathfrak{sl}_2(\mathbb{F})$-module via the adjoint representation. Decompose M into irreducible $\mathfrak{sl}_2(\mathbb{F})$-modules and show that $M \cong V(0) \oplus V(1) \oplus V(1) \oplus V(2)$ as $\mathfrak{sl}_2(\mathbb{F})$-modules.

4. Suppose (just for this exercise) that $\mathbb{F}$ has characteristic $p > 0$. What numbers can occur as p? Show that the $\mathfrak{sl}_2(\mathbb{F})$-module $V(m)$ in Exercise 1 is irreducible if $m < p$, but reducible when $m = p$.

5. Let $\lambda \in \mathbb{F}$ be an arbitrary scalar. Let $M(\lambda)$ be a vector space with a countably infinite basis $v_0, v_1, v_2, \ldots$. Define $Hv_i = (\lambda - 2i)v_i$ and $Yv_i = (i + 1)v_{i+1}$ and $Xv_i = (\lambda - i + 1)v_{i-1}$ where $v_{-1} = 0$. Your solution to Exercise 1 should easily extend to an argument that that these formulas make $M(\lambda)$ into an $\mathfrak{sl}_2(\mathbb{F})$-module. For which values of λ is $M(\lambda)$ irreducible? Prove your answer.

6. Assume L is a classical linear Lie algebra of type A_n. Prove that the set H of all diagonal matrices in L is a maximal toral subalgebra.

7. Assume L is a classical linear Lie algebra of type A_n. Determine the roots and root spaces corresponding to the root space decomposition of L relative to the maximal toral subalgebra of diagonal matrices H.

8. Assume L is a classical linear Lie algebra of type C_n. Prove that the set H of all diagonal matrices in L is a maximal toral subalgebra.

9. Assume L is a classical linear Lie algebra of type C_n. Determine the roots and root spaces corresponding to the root space decomposition of L relative to the maximal toral subalgebra of diagonal matrices H.