MATH 5143 – Lecture #7

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Last week: Cartan's criterion for solvability

(All vector spaces / Lie algebras mentioned are finite-dim and defined over an algebraically closed field $\mathbb{F}$ of characteristic zero (e.g. $\mathbb{C}$))

Let L be a Lie algebra. Recall that $L^{(0)} := L$ and $L^{(n+1)} := [L^{(n)}, L^{(n)}]$ and L is solvable if $L^{(n)} = 0$ for $n \gg 0$.

Thm Suppose $L \subseteq \mathfrak{gl}(V)$ where V is a fin. dim. vector space. If $\operatorname{trace}(XY) = 0$ $\forall X \in [L, L]$ $\forall Y \in L$ then L is solvable

<u>Cor</u> If $\operatorname{trace}(\underbrace{\operatorname{ad} X}_{} \operatorname{ad} Y) = 0 \ \forall X \in [L,L]$
$\forall Y \in L$

then L is solvable

(this is the linear map $L \to L$
$Z \mapsto [X,Z]$)

Killing form on L: this is the symmetric bilinear form $\kappa(X,Y) \overset{\text{def}}{=} \operatorname{trace}(\operatorname{ad} X \operatorname{ad} Y)$

Special property of κ: it is <u>associative</u> meaning

$$\kappa([X,Y],Z) = \kappa(X,[Y,Z])$$

$$\forall X,Y,Z \in L$$

Prop The Killing form for any ideal of L is the restriction of the Killing form of L.

The **radical** of $\mathcal{K}$ is $\mathrm{Rad}(\mathcal{K}) = \{x \in L \mid \mathcal{K}(x,y) = 0 \ \forall y \in L\}$

The Killing form is **nondegenerate** if $\underbrace{\mathrm{Rad}(\mathcal{K}) = 0}$.

this condition can be checked by computing $\det[\mathcal{K}(x_i, x_j)]$ for a basis $\{x_i\}$ for L

The **radical** of L is its (unique) maximal solvable ideal, denoted $\mathrm{Rad}(L)$.

$\Leftrightarrow$ L has no nonzero solvable ideals

$\Leftrightarrow$ L has no nonzero abelian ideals

The Lie algebra L is **semisimple** if $\mathrm{Rad}(L) = 0$

Thm $\mathrm{Rad}(\mathcal{K}) = 0$ if and only if $\mathrm{Rad}(L) = 0$

Thm L is semisimple if and only L has simple ideals $L_1, L_2, \ldots, L_n$ such that

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_n \text{ (as Lie algebras)}$$

In this case, every ideal of L is a direct sum $L_{i_1} \oplus L_{i_2} \oplus \cdots \oplus L_{i_k}$ for some $i_1 < i_2 < \cdots < i_k$

Cor If L is semisimple then $L = [L, L]$ and all ideals as well as all homomorphic images of L are also semisimple.

Rest of today: basic concepts in representation theory of Lie algebras

Terminology Throughout, L is a semisimple Lie algebra

An **L-representation** is a Lie algebra morphism

$$\phi : L \to \mathfrak{gl}(V) \quad \text{(for some vector space } V\text{)}$$

Explicitly: ϕ is a linear map with $\phi([x,y]) = [\phi(x), \phi(y)]$

An **L-module** is a vectorspace V with a bilinear operation

$$L \times V \to L \qquad (x, v) \mapsto x \cdot v$$

such that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) \quad \forall x, y \in L, v \in V$$

L-reprs and L-modules are equivalent notions, just different syntax

Equivalent in this sense:

Prop If $\phi : L \to gl(V)$ is an L-repn then V is an L-module for the action $X \cdot v \overset{\text{def}}{=} \phi(X)(v)$ for $X \in L$, $v \in V$

Pf The action is bilinear and we have

$$X \cdot (Y \cdot v) - Y \cdot (X \cdot v) = \phi(X)\big(\phi(Y)(v)\big) - \phi(Y)\big(\phi(X)(v)\big)$$

$$= \big(\phi(X)\phi(Y) - \phi(Y)\phi(X)\big)(v) = [\phi(X), \phi(Y)](v)$$

$$= \phi([X,Y])(v) = [X,Y] \cdot v \quad \forall X, Y \in L, \ v \in V \quad \square$$

Prop If V is an L-module then the map $\phi : L \to gl(V)$ defined by $\phi(X) : v \mapsto X \cdot v$ for $v \in V$ is an L-repn.

Pf Similar straightforward algebra to check. $\square$

Suppose V is an L-module.

A <u>submodule</u> of V is a subspace $U \subseteq V$ such that $X \cdot u \in U \ \forall X \in L, \forall u \in U$.

A <u>morphism</u> of two L-modules V and W is a linear map $f: V \to W$ such that

$$f(X \cdot v) = X \cdot f(v) \quad \forall v \in V, X \in L.$$

The <u>kernel</u> of a morphism $f: V \to W$ is a submodule: $\mathrm{Ker}(f) \overset{\text{def}}{=} \{v \in V \mid f(v) = 0\}$.

If an L-module morphism $f: V \to W$ is a bijection then f is an isomorphism.

An L-module V is irreducible if its only L-submodules are 0 and $V \neq 0$. [meaning V has exactly two submodules]

Zero modules are not considered irreducible because we want a unique direct sum decomp into irreducible submodules

V is completely reducible if there are irreducible L-submodules $V_i \subseteq V$ such that $V = \bigoplus_i V_i$

Here $\oplus$ refers to obvious notion of direct sum for L-modules

Fundamental result (state without proof):

(Schur's lemma) Suppose $\phi : L \to \mathfrak{gl}(V)$ is an irreducible L-repn (meaning that the associated L-module structure on V is irreducible). Then the only linear maps $f : V \to V$ with

$$f \circ \phi(X) = \phi(X) \circ f \quad \forall\, X \in L$$

are the scalar maps $f_c : V \to V$, $v \mapsto cv$ for fixed $c \in \mathbb{F}$.

(Requires $\mathbb{F}$ to be algebraically closed, characteristic zero)

Dual / contragradient of an L-module

Suppose V is an L-module. Define $V^* = \left\{ \begin{array}{c} \text{linear maps} \\ V \to \mathbb{F} \end{array} \right\}$

Fact V^* is an L-module for the action

$$X \cdot f = \left(\begin{array}{c} \text{the linear map } V \to \mathbb{F} \\ \text{sending } v \mapsto -f(X \cdot v) \end{array} \right) \quad \text{for } f \in V^*$$

Pf For $X, Y \in L$, $f \in V^*$, $v \in V$ we have

$$([X,Y] \cdot f)(v) = -f([X,Y] \cdot v) = -f(X \cdot (Y \cdot v) - Y \cdot (X \cdot v))$$

$$= -f(X \cdot (Y \cdot v)) + f(Y \cdot (X \cdot v)) = (X \cdot f)(Y \cdot v) - (Y \cdot f)(X \cdot v)$$

$$= -(Y \cdot (X \cdot f))(v) + (X \cdot (Y \cdot f))(v) = (X \cdot (Y \cdot f) - Y \cdot (X \cdot f))(v) \quad \square$$

Tensor products of L-modules

Suppose V and W are L-modules, say with bases $\{v_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$

has basis $\{v_i \otimes w_j\}_{(i,j)\in I\times J}$

The tensor product $V \otimes W$ is the vector space spanned by all tensors $v \otimes w$ $(v\in V,\ w\in W)$ where

$(v+v')\otimes w = v\otimes w + v'\otimes w$ and $v\otimes(w+w') = v\otimes w + v\otimes w'$

and $(av)\otimes w = v\otimes(aw)$ for $a\in\mathbb{F}$, $v,v'\in V$, $w,w'\in W$

Fact $V\otimes W$ is an L-module for the action

$$x\cdot(v\otimes w) = (x\cdot v)\otimes w + v\otimes(x\cdot w)$$ for $x\in L$, $v\in V$, $w\in W$

Pf $[X,Y]\cdot(v\otimes w) = ([X,Y]\cdot v)\otimes w + v\otimes([X,Y]\cdot w)$

$$= \boxed{X\cdot(Y\cdot v)\otimes w - Y\cdot(X\cdot v)\otimes w + v\otimes X\cdot(Y\cdot w) - v\otimes Y\cdot(X\cdot w)}.$$

But we also have $X\cdot(Y\cdot(v\otimes w)) - Y\cdot(X\cdot(v\otimes w))$

$$= \boxed{X\cdot(Yv\otimes w + v\otimes Yw) - Y\cdot(Xv\otimes w + v\otimes Xw)}$$

equal

and after some cancellation this is same as above. $\square$

<u>Exercise</u>. The linear map $V^*\otimes V \to \mathfrak{gl}(V)$

$f\otimes w \mapsto$ (the linear map $V\to V$ sending $v\mapsto f(v)w$)

is an isomorphism of vector spaces if $\dim V<\infty$.

The L-module structure on $\mathfrak{gl}(V)$ making this map a module isomorphism is $X\cdot f =$ (the linear map $V\to V$ sending $v\mapsto X\cdot f(v) - f(X\cdot v)$) for $X\in L$, $f\in\mathfrak{gl}(V)$