MATH 5143 – Lecture #8

Throughout: L is semisimple Lie algebra over algebraically closed field $\mathbb{F}$ of characteristic zero, $\dim L < \infty$

Last time: basic concepts in representation theory of Lie algebras

Goals for today: discuss Casimir element as a tool for proving Weyl's thm: fin. dim. repns of semisimple L are "completely reducible"

Also assume V is an $\mathbb{F}$-vector space with $\dim V < \infty$

Recall:

$\boxed{\text{Schur's lemma}}$ Suppose $\phi : L \to \mathfrak{gl}(V)$ is an irreducible L-repn (meaning that the associated L-module structure on V is irreducible). Then the only linear maps $f : V \to V$ with

$$f \circ \phi(X) = \phi(X) \circ f \quad \forall\, X \in L$$

are the scalar maps $f_c : V \to V$, $v \mapsto cv$ for fixed $c \in \mathbb{F}$.

Casimir element

An L-repn $\phi : L \to \mathfrak{gl}(V)$ is faithful if $\ker \phi = 0$, meaning ϕ is injective.

Assume $\phi : L \to \mathfrak{gl}(V)$ is a faithful L-repn.

Define $\beta : L \times L \to \mathbb{F}$
$$(X, Y) \mapsto \beta(X, Y) \overset{\text{def}}{=} \operatorname{trace}_V(\phi(X)\phi(Y))$$

This bilinear form is symmetric and associative:

$$\beta([X, Y], Z) = \beta(X, [Y, Z]) \quad \forall X, Y, Z \in L$$

The Killing form of L is β for $\phi = \operatorname{ad} : L \to \mathfrak{gl}(L)$

The radical of β is the ideal $S = \{X \in L \mid \beta(X, Y) = 0 \ \forall Y \in L\}$

for L

In fact, S is a solvable ideal of L, since $S \cong \phi(S)$ (by faithfulness of ϕ) and Cartan's criterion holds for $\phi(S)$ $\left[\underbrace{\text{trace}(XY)}_{\text{this is equal}} = 0 \;\; \forall X \in \phi([S,S]) \;\; \forall Y \in \phi(S) \right]$

this is equal to β evaluated on elems of $S \times L$

We are assuming that L is semisimple, so

we conclude that: Prop The form

$$\beta(x,y) = \text{trace}(\phi(x)\phi(y))$$

is nondegenerate (ie. $S = 0$)

Conversely, suppose $\beta : L \times L \to \mathbb{F}$ is any symmetric, associative, nondegenerate bilinear form.

Choose a basis $x_1, x_2, \ldots, x_n$ for L and define $y_1, y_2, \ldots, y_n$ as the (unique) dual basis with

$$\beta(x_i, y_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Fix $z \in L$ and define $a_{ij}, b_{ij} \in \mathbb{F}$ such that

$$[z, x_i] = \sum_j a_{ij} x_j \quad \text{and} \quad [z, y_i] = \sum_j b_{ij} y_j$$

Lemma $\boxed{a_{ik} = -b_{ki}}$ $\forall\, i, k$

follows by associativity and bilinearity

Pf $a_{ik} = \sum_j \beta(x_j, y_k) a_{ij} = \beta([z, x_i], y_k) = \beta(x_i, -[z, y_k])$

$= -\sum_j \beta(x_i, y_j) b_{kj} = -b_{ki}$ □

Suppose $\phi : L \to \mathfrak{gl}(V)$ is an L-repn. [β is given as above]

Define $c_\phi(\beta) \stackrel{\text{def}}{=} \sum_i \phi(x_i)\phi(y_i) \in \mathfrak{gl}(V)$

[there are dual bases defined wrt β
the def here looks like it depends on the
choice of these bases]

Prop Then $[\phi(z), c_\phi(\beta)] = 0 \ \forall z \in L$.

So (multiplication by) $c_\phi(\beta)$ is a linear map $V \to V$
that commutes with the action of L via ϕ.

Pf $[\phi(z), c_\phi(\beta)] = \sum_i [\phi(z), \phi(x_i)]\phi(y_i) + \sum_i \phi(x_i)[\phi(z), \phi(y_i)]$

$= \sum_{i,j} (\underbrace{a_{ij} + b_{ji}}_{=0}) \phi(x_i)\phi(y_j) = 0$. $\square$

Def When $\phi : L \to \mathfrak{gl}(V)$ is a faithful L-repn

we define the **Casimir element** to be

as given on previous slide, requires a choice of basis + dual basis for L

$$c_\phi = \overbrace{c_\phi(\beta)}^{} \in \mathfrak{gl}(V)$$ for form $\beta(x,y) = \text{trace}(\phi(x)\phi(y))$

[This makes sense since we already checked that this form is nondegenerate + associative.]

Two key facts ① $\boxed{\text{trace}(c_\phi) = \dim L}$

as a linear map $V \to V$

Pf $\text{trace}(c_\phi) = \sum_i \text{trace}(\phi(x_i)\phi(y_i)) = \sum_i \underbrace{\beta(x_i, y_i)}_{=1} = \dim L$ $\square$

② If ϕ is irreducible then $\boxed{c_\phi = \frac{\dim L}{\dim V}} \in \mathbb{F} \subseteq \mathfrak{gl}(V)$

Pf If ϕ is irreducible then Schur's lemma implies that c_ϕ is a scalar since $[\phi(x), c_\phi] = 0 \ \forall x \in L$. $\square$ [So if V is irreducible, c_ϕ does not depend on the choice of basis for L]

Ex Let $L = sl_2(\mathbb{F}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a+d=0 \right\}$

L has basis $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

Suppose $V = \mathbb{F}^2$ and $\phi : L \to gl(V)$ is identity map.

Basis dual to $\boxed{X, H, Y}$ in $\underset{L}{\underline{\text{traceform}}}$ is $\boxed{Y, \tfrac{1}{2}H, X}$

$\beta(A,B) = \text{trace}(AB)$ since $\phi = id$

so $c_\phi = XY + \frac{1}{2}H^2 + YX = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3/2 & 0 \\ 0 & 3/2 \end{bmatrix} \leftarrow$ a scalar matrix.

When L is semisimple but $\phi : L \to \mathfrak{gl}(V)$ is <u>not</u> faithful, we define $c_\phi \in \mathfrak{gl}(V)$ to be the Casimir element of the faithful repn $\phi : L/\ker\phi \to \mathfrak{gl}(V)$.

Lemma Let $\phi : L \to \mathfrak{gl}(V)$ be an L-repn, with L semisimple. Then $\phi(L) \subseteq \underbrace{\mathfrak{sl}(V)}_{\text{traceless endomorphisms}} \subseteq \mathfrak{gl}(V)$. [Thus if $\dim V = 1$ then $\phi(L) = 0$ as $\mathfrak{sl}(V) = 0$]

<u>Pf</u> We have $L = [L, L]$ by semisimplicity so

$$\phi(L) = \phi([L, L]) = [\phi(L), \phi(L)] \subseteq [\mathfrak{gl}(V), \mathfrak{gl}(V)] = \mathfrak{sl}(V)$$

$\square$

Thm (Weyl's theorem) Suppose $\phi : L \to \mathfrak{gl}(V)$ is an L-repn. As usual, we assume L is semisimple and $\dim V < \infty$.

Then ϕ is completely reducible, meaning that there are irreducible L-submodules $V_1, V_2, \dots, V_n \subseteq V$ such that

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$$

Pf By replacing L by $L/\ker\phi$, we may assume that ϕ is faithful.

Suppose $W \leq V$ is a proper L-submodule. By induction on dimension, we just need to show that there is a complementary L-submodule U with

$$V = W \oplus U$$

(we can find a subspace U such that direct sum holds as vector spaces, hard part is to find a submodule.)

Three steps to proof. First step:

① Assume W is irreducible and $\dim V = \dim W + 1$ so $V = W \oplus \mathbb{F}$ as a vector space

Let $c = c_\phi$ be Casimir element.

Then $v \mapsto cv$ is an L-module endomorphism as c commutes with $\phi(L)$.

Thus $cW \leq W$ and $\underbrace{\ker(c) \overset{\text{def}}{=} \{v \in V \mid cv = 0\} \text{ is an } L\text{-submodule}}_{\text{because } c \text{ commutes } w/\, \phi(L)}$

↑ because W is a submodule and $c \in \langle \phi(L) \rangle \leq \mathfrak{gl}(V)$

L acts trivially on $V/W = \mathbb{F}$ because all 1-dim repns of semisimple Lie algebras are trivial (by lemma). This means $c(V/W) = 0 \Rightarrow cV \leq W$.

Therefore $\boxed{\dim \ker(c) \geq 1}$, since otherwise $cV = V$.

But c acts on W as a scalar by Schur's lemma, and this scalar cannot be zero since $\mathrm{trace}_W(c) = \mathrm{trace}_V(c) = \dim L \neq 0$. Therefore $\ker(c) \cap W = 0$ since c acts as nonzero scalar on W. As $\dim \ker c + \dim W \geq \dim V$ we conclude that $V = W \oplus \ker c$.

both L-submodules

(2) Suppose $V = W \oplus \mathbb{F}$ as vector spaces (so $\dim V = \dim W + 1$)
but W is <u>not</u> irreducible as an L-module.

Then there is a nonzero proper submodule $W' \subset W$
and by induction $V/W' = W/_{W'} \oplus \tilde{U}$ for some
as L-modules.
L-submodule $\tilde{U} \subset V/W'$.

Define U to be preimage of $\tilde{U}$ under quotient map $V \to V/W'$.
Then U is an L-submodule containing W' (so $\tilde{U} = U/W'$)

By induction $U = W' \oplus W''$ for some L-module W'' (as L-modules)
and then $V = W \oplus W''$ since $\dim W + \dim W'' = \dim V$
as L-modules (as $\dim(V/W') = \dim(W/W') + \dim(U/W')$)

and $W \cap W'' = 0$ (as $W \cap W'' \subseteq W'$ but $W' \cap W'' = 0$)

③ Finally suppose W is arbitrary and $\dim(V/W) \geq 1$.

Let $\mathrm{Hom}(V, W)$ be L-module of linear maps $f: V \to W$ with L acting as $X \cdot f : v \mapsto X \cdot f(v) - f(X \cdot v)$

Let $A = \{ f \in \mathrm{Hom}(V,W) \mid f|_W \text{ is a scalar map} \}$

$B = \{ f \in \mathrm{Hom}(V,W) \mid f|_W = 0 \}$

Then A and B are L-submodules with $L \cdot A \subseteq B$ since if $f(w) = aw \; \forall w \in W$ where $a \in \mathbb{F}$ then

$$(X \cdot f)(w) = X \cdot \underbrace{f(w)}_{=aw} - f(\underbrace{X \cdot w}_{\in W}) = aX \cdot a - aX \cdot w = 0$$

But $\dim(A) = \dim(B) + 1$ since if $f, g \in A$ then $af + bg \in B$ for some $a, b \in \mathbb{F}$

Therefore $A = B \oplus C$ for some L-submodule C with $\dim C = 1$ (by case ②). Suppose $C = \mathbb{F}\text{-span}\{h\}$ for some $h : V \to W$. We may assume $h|_W = \mathrm{id}$ (after rescaling).

<u>Claim</u>: $\ker(h) = \{v \in V \mid h(v) = 0\}$ is an L-submodule and $V = W \oplus \ker(h)$ as L-modules

<u>Pf</u>: If $h(v) = 0$ and $x \in L$ then

$$h(x \cdot v) = -(\underbrace{x \cdot h}_{\text{scalar multiple of } h})(v) + x \cdot h(v) = -0 + 0 = 0.$$

Now observe $V = \operatorname{image}(h) \oplus \ker(h) = W \oplus \ker(h)$ $\square$

↖ for any linear $h : V \to W$