# MATH 5143 - Lecture #9

# Math 5143 - Lecture 5

Last time: representations for (semisimple) Lie algebras

$L$ is a Lie algebra of finite dimension over an alg. closed field $\mathbb{F}$ of char. zero.

An <u>$L$-repn</u> is a Lie algebra morphism $\phi : L \to \mathfrak{gl}(V)$ (for some vector space $V$, which could have $\dim V = \infty$)

An <u>$L$-module</u> is a vector space $V$ with a bilinear multiplication
$$L \times V \to V$$
$$(x, v) \mapsto xv \text{ or } x \cdot v$$
such that

$$[x, y]v = x(yv) - y(xv) \quad \forall\, x, y \in L,\ v \in V$$

Assume $L$ is <u>semisimple</u> $\Leftrightarrow$ ($L$ has no solvable ideals)

Recall: this means that $L \cong L_1 \oplus L_2 \oplus \cdots \oplus L_n$ where each $L_i$ is a $\underbrace{\text{simple}}$ ideal. Thus $Z(L) = 0$ where $Z(L) = \{x \in L \mid \operatorname{ad} x = 0\}$.

non-abelian, with no nonzero proper ideals

<u>Weyl's theorem</u> Every finite-dimensional $L$-module is completely reducible. Also if $\phi : L \to \mathfrak{gl}(V)$ is any $\begin{pmatrix}\text{finite. dim.}\\ L\text{-repn}\end{pmatrix}$ then $\phi(L) \subseteq \mathfrak{sl}(V) \subsetneq \mathfrak{gl}(V)$.

<u>Cor</u> If $\dim V = 1$ and $\phi : L \to \mathfrak{gl}(V)$ then $\phi(L) = 0$

since $\mathfrak{sl}(V) = 0$ if $\dim V = 1$.

As we assume $L$ is semisimple, we have $Z(L)=0$ and the adjoint repn $\mathrm{ad}: L \to \mathfrak{gl}(L)$ is faithful.

Define the <u>abstract Jordan decomposition</u> of $x \in L$ to be $x = x_s + x_n$ where $x_s, x_n \in L$ are the unique elements with
$$\mathrm{ad}(x_s) = (\mathrm{ad}\, x)_s \text{ and}$$
$$\mathrm{ad}(x_n) = (\mathrm{ad}\, x)_n$$

This definition is ambiguous if it already holds that $L \subseteq \mathfrak{gl}(V)$ for some $V$.

this is defined using Jordan decomposition of elements of $\mathfrak{gl}(V)$ (here $V = L$)

<u>Thm</u> If $L \leq \mathfrak{gl}(V)$ then the components $x_s$ and $x_n$ of the usual Jordan decomposition of $x \in L$ are both contained in $L$, and they coincide with the components of the abstract Jordan decomposition of $x$.

[second claim is a consequence of first, via the properties defining both decompositions

First claim is nontrivial because although we know $x_s$ and $x_n$ are polynomials in $x$, $L$ is not a subalgebra of $\mathfrak{gl}(V)$.]

Pf sketch $V$ is an $L$-module since $L \subseteq gl(V)$

For each $L$-submodule $W \subseteq V$ define

$$L_W = \{ y \in gl(V) \mid yW \subseteq W \text{ and } \mathrm{trace}_W(y) = 0 \}$$

Since $L = [L, L]$, we have $L \subseteq L_W$.

Define $L' = \bigcap_{W \text{ submodule } V} L_W \cap N_{gl(V)}(L)$

$N_{gl(V)}(L) = \{ y \in gl(V) \mid [y, L] \subseteq L \}$

let $x_s$, $x_n$ be components of Jordan decomp of $x$, these are in $gl(V)$

Fix $x \in L$. Since $x_s$ and $x_n$ are polynomials in $L$, and as $(\mathrm{ad}\, x)(L) \subseteq L$, we have $x_s, x_n \in N_{gl(V)}(L)$

Also $x_s, x_n \in L_W$ for all $W$. So it suffices to show $L = L'$.

Showing that $L = L'$ is a consequence of Weyl's theorem $\rightsquigarrow$ (see textbook) $\square$

<u>Thm</u> If $L$ is semisimple and $\phi : L \to \mathfrak{gl}(V)$ is an $L$-repn with $\dim V < \infty$, then for any $x \in L$ with the abstract Jordan decomposition $x = x_s + x_n$ the usual Jordan decomp of $\phi(x)$ is $\phi(x) = \phi(x_s) + \phi(x_n)$

[we saw this earlier for $\phi = \mathrm{ad}$]

Pf sketch of thm. We already know this holds if $\phi = \mathrm{id}$ [base case]

For general $\phi$, observe that $\phi(L)$ has a basis of eigenvectors for $\operatorname{ad}\phi(X_s)$ since $L$ does for $\operatorname{ad}(X_s)$. (In particular we have

$$\operatorname{ad}\phi(X_s)(\phi(Y)) = [\phi(X_s), \phi(Y)] = \phi([X_s, Y]) = \phi((\operatorname{ad} X_s)(Y)))\,.$$

Therefore $\operatorname{ad}\phi(X_s)$ is semisimple. Likewise $\operatorname{ad}\phi(X_n)$ is nilpotent. But $\phi(L)$ is semisimple so base case $\Rightarrow$ $\phi(X)_s = \phi(X_s)$ and $\phi(X)_n = \phi(X_n)$ $\square$

(this is because we also have $[\phi(X_s), \phi(X_n)] = \phi([X_s, X_n]) = 0$

Next: representations of $\mathfrak{sl}_2(\mathbb{F}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : \begin{array}{l} a,b,c,d \in \mathbb{F} \\ a+d=0 \end{array} \right\}$

Let $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Then $\mathfrak{sl}_2(\mathbb{F}) = \mathbb{F}\text{-span}\{X, H, Y\}$ and

$$\boxed{\begin{aligned} [H,X] &= 2X \\ [H,Y] &= -2Y \\ [X,Y] &= H \end{aligned}}$$

Consider an arbitrary $\mathfrak{sl}_2(\mathbb{F})$-module $V$ of finite dimension. Since $H$ is semisimple $\equiv$ (diagonalizable in adjoint repn), the thm just proved says that $V$ must have a basis of eigenvectors for $H$.

This property relies on $\mathbb{F}$ being algebraically closed, so that all eigenvalues for $H$ are present.

Key takeaway: we may decompose

$$V = \bigoplus_{\substack{\text{eigenvalues} \\ \lambda \text{ for } H}} V_\lambda \quad \text{where } V_\lambda = \{ v \in V \mid Hv = \lambda v \}$$

Note that our def of $V_\lambda$ makes sense even when $\lambda$ is not an eigenvalue for $H$, but in that case $V_\lambda = 0$.

We refer to the eigenvalues of $H$ as <u>weights</u> and the nonzero subspaces $V_\lambda$ as <u>weight spaces</u>

$X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

<u>Lemma</u> If $v \in V_\lambda$ and $Xv \in V_{\lambda+2}$ and $Yv \in V_{\lambda-2}$

<u>Pf</u> $HXv = \underbrace{[H,X]}_{=2X}v + XHv = 2Xv + X\lambda v = (\lambda+2)Xv$

Argument to show $HYv = (\lambda-2)Yv$ is similar. $\square$

Assume our $\mathfrak{sl}_2(\mathbb{F})$-module $V$ has $0 < \dim V < \infty$.

There must exist at least one $\lambda \in \mathbb{F}$ with $V_\lambda \neq 0 = V_{\lambda+2}$

$\uparrow$ as $V \neq 0$ $\quad$ $\uparrow$ as $\dim V < \infty$

For this $\lambda$, we have $\underbrace{Xv}_{\in V_{\lambda+2} = 0} = 0$ for all $v \in V_\lambda$

We call the nonzero elements of this $V_\lambda$ maximal weight vectors of $V$ with weight $\lambda$.

Lemma Assume $V$ is irreducible $\mathfrak{sl}_2(\mathbb{F})$-module.

Choose a maximal weight vector $v_0 \in V_\lambda$

Define $v_{-1} = 0$ and $v_k = \frac{1}{k!} Y^k v_0$ for $k \geq 0$. Then:

(a) $Hv_i = (\lambda - 2i)v_i$ $\quad$ pf: apply prev. lemma since $v_i \in V_{\lambda-2i}$

(b) $Yv_i = (i+1)v_{i+1}$ $\quad$ pf: by definition

(c) $Xv_i = (\lambda - i + 1)v_{i-1}$ $\quad$ pf: by induction using formulas for Lie brackets + (a)(b)

Continue to assume $V$ is irreducible, $\dim V < \infty$.

Since the nonzero $v_k$'s are $H$-eigenvectors with distinct eigenvalues, they are linearly independent

There exists a smallest $m$ with $v_m \neq 0 = v_{m+1} = v_{m+2} = \cdots$.

Then must have $V = \mathbb{F}\text{-span}\{v_0, v_1, \ldots, v_m\}$.

irred.

is a submodule by prev lemma

hence this is equality

In the basis $v_0\, v_1 \ldots v_m$ for $V$ the matrices of

$H, X, Y$ are diagonal, strictly upper-$\Delta$, and strictly lower-$\Delta$

Moreover: $0 = X0 = Xv_{m+1} \underset{\text{by lemma}}{=} (\lambda - m)v_m$.

__Cor.__ Thus $\lambda = m \in \mathbb{Z}_{\geq 0}$ and the weight of any highest weight vector in an [irred. fin. dim.] $\mathfrak{sl}_2(\mathbb{F})$-module is a nonnegative integer, called the **highest weight**.

__Thm__ Let $V$ be an irreducible $\mathfrak{sl}_2(\mathbb{F})$-repn. with $\dim V = m+1 < \infty$.

implicit in above discussion

(a) Then $V = V_{-m} \oplus V_{-m+2} \oplus V_{-m+4} \oplus \cdots \oplus V_{m-2} \oplus V_m$

where each $V_i = \{v \in V \mid Hv = iv\} \neq 0$.

(b) $V$ has a unique highest weight space of weight $m$

(c) For each $m \geq 0$, there exists [exactly one] irreducible $\mathfrak{sl}_2(\mathbb{F})$-module of dimension $m+1$, up to isomorphism. → to prove: check that the formulas for action of $X, Y, H$ in previous lemma define an $\mathfrak{sl}_2(\mathbb{F})$-module

Note that if $m = \dim V - 1$ is odd then $V$ looks like

$$V_{-m} \xrightarrow{Y} V_{-m+2} \xrightarrow{Y} \cdots \xrightarrow{Y} V_{-2} \xrightarrow{Y} V_0 \xrightarrow{Y} V_2 \xrightarrow{Y} \cdots \xrightarrow{Y} V_{m-2} \xrightarrow{Y} V_m$$

while if $m$ is even, $V$ looks like

$$V_{-m} \xrightarrow{Y} V_{-m+2} \xrightarrow{Y} V_{-3} \xrightarrow{Y} V_{-1} \xrightarrow{Y} V_1 \xrightarrow{Y} V_3 \xrightarrow{} \cdots \xrightarrow{Y} V_{m-2} \xrightarrow{Y} V_m$$

and of dim 1

So exactly one of $V_0$ or $V_1$ is nonzero when $V$ is irreducible.

Cor If $V$ is any finite-dim. $\mathfrak{sl}_2(\mathbb{F})$-module then the eigenvalues for $H \in \mathfrak{sl}_2(\mathbb{F})$ acting on $V$ are integers, and if $\lambda$ is one of these eigenvalues then so is $-\lambda$. Also, if $V_i = \{v \in V \mid Hv = iv\}$ then the number of $\boxed{\text{summands}}$ in any irreducible decomposition of $V$ is $\boxed{\dim V_0 + \dim V_1}$

[Pf : $\mathfrak{sl}_2(\mathbb{F})$ is semisimple, so we just apply Weyl's theorem.]