Math 5143 - Lecture 13

Root space decomposition

Let L be a nonzero, finite dim. semisimple Lie alg.

A subalgebra $H \subseteq L$ is toral if every $x \in H$ is semisimple. Any toral subalgebra is abelian ⊛

Choose a maximal toral subalgebra $H \subseteq L$.

For this maximal toral subalgebra $H \subseteq L$, the corresponding root space decomposition is $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ where

Φ is a finite subset of H^*, $L_\alpha \overset{\text{def}}{=} \{x \in L \mid [h,x] = \alpha(h)x \ \forall h \in H\}$

$0 \notin \Phi$ since $H = L_0$. Existence of this decomp. is clear from ⊛

Call L_α a root space and $\alpha \in \Phi$ a root

Prop (a) If $\alpha, \beta \in \Phi$ then $\beta(H_\alpha) \in \mathbb{Z}$ call this a Cartan integer

and $\beta - \beta(H_\alpha)\alpha \in \Phi$

(b) If $\alpha, \beta \in \Phi$, and $\alpha + \beta \in \Phi$, then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$

(c) If $\alpha, \beta \in \Phi$ and $\alpha + \beta \neq 0$ then there are integers $r, q \geq 0$ such that

$$(\beta + \mathbb{Z}\alpha) \cap \Phi = \{\beta + i\alpha \mid i \in \mathbb{Z}, -r \leq i \leq q\}$$

"no gaps in the α-root string through β"

Also, it holds that $\boxed{\beta(H_\alpha) = r - q}$

(d) L is generated by the root spaces L_α ($\alpha \in \Phi$) as a Lie algebra.

Pf will show that (c) holds. Other properties are straightforward.

Pf Set $K = \sum_{i \in \mathbb{Z}} \underbrace{L_{\beta+i\alpha}}_{\text{zero for all but finitely many } i \in \mathbb{Z}}$ where $\alpha, \beta \in \Phi$ with $\alpha+\beta \neq 0$.

No multiple of α except $\pm\alpha$ is a root, so we have $\beta + i\alpha \neq 0$ $\forall i \in \mathbb{Z}$. K is a submodule of $S_\alpha \cong sl_2(\mathbb{F})$ and each $L_{\beta+i\alpha}$ is either zero if $\beta + i\alpha \notin \Phi$ or 1-dimensional if $\beta + i\alpha \in \Phi$ in which case $(\beta+i\alpha)(H_\alpha) = \beta(H_\alpha)+2i$

(recall that def of H_α gives $\alpha(H_\alpha)=2$)

In latter case, $\beta(H_\alpha)+2i$ is the weight of H_α on $L_{\beta+i\alpha}$.

Because all of these weights differ by an even integer, exactly one of the numbers 0 or 1 can occur as a weight, so K is an irreducible S_α-module. Thus if r, q are maximal with $\beta - r\alpha \in \Phi$, $\beta + q\alpha \in \Phi$ then the corresponding weights $\beta(H_\alpha)-2r$ and $\beta(H_\alpha)+2q$ sum to zero, and (c) follows. □

We have our root space decomp $L = H \oplus \bigoplus_{\alpha \in \Phi \subseteq H^* \setminus 0} L_\alpha$

and $\kappa|_{H \times H}$ is nondegenerate, and we defined $t_\alpha \in H$ for $\alpha \in H^*$ to have $\kappa(t_\alpha, h) = \alpha(h)\ \forall h \in H$.

We now further define $(\alpha, \beta) \overset{\text{def}}{=} \kappa(t_\alpha, t_\beta)$ for $\alpha, \beta \in H^*$.

Let $E_{\mathbb{Q}} = \mathbb{Q}\text{-span}\{\alpha \in \Phi\}$, and $E = \mathbb{R} \otimes_{\mathbb{Q}} E_{\mathbb{Q}}$

One can show that:

<u>Thm</u> $(\cdot, \cdot)$ restricts to a $\underbrace{\text{positive definite}}_{\text{means } (\alpha,\alpha) > 0 \text{ for } 0 \neq \alpha \in H^*}$ form on E with $(\alpha, \beta) \in \mathbb{Q}\ \forall \alpha, \beta \in \Phi$. Additionally:

① Φ spans E ② If $\alpha \in \Phi$ then $\mathbb{R}\alpha \cap \Phi = \{-\alpha, \alpha\}$

③ If $\alpha, \beta \in \Phi$ then $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$ ④ $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}\ \forall \alpha, \beta \in \Phi$

__Ex__ If $L = \mathfrak{sl}_n(\mathbb{F})$ and $H = $ (diagonal matrices in L)

then $\Phi = \{ \varepsilon_i - \varepsilon_j \mid 1 \le i, j \le n, \ i \ne j \}$

where $\varepsilon_i : H \to \mathbb{F}$
$D \mapsto D_{ii}$ (diagonal entry in row i)

As noted earlier, we have $t_{\varepsilon_i - \varepsilon_j} = \frac{1}{4}(E_{ii} - E_{jj})$

and $(\varepsilon_i - \varepsilon_j, \varepsilon_k - \varepsilon_l) \overset{\text{def}}{=} \kappa(t_{\varepsilon_i - \varepsilon_j}, t_{\varepsilon_k - \varepsilon_l})$

$$= \frac{1}{4} \langle \varepsilon_i - \varepsilon_j, \varepsilon_k - \varepsilon_l \rangle$$

where $\langle \varepsilon_i, \varepsilon_j \rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$. So $2\dfrac{(\varepsilon_i - \varepsilon_j, \varepsilon_k - \varepsilon_l)}{\underbrace{(\varepsilon_k - \varepsilon_l, \varepsilon_k - \varepsilon_l)}_{= 1/2}} = \underbrace{\langle \varepsilon_i - \varepsilon_j, \varepsilon_k - \varepsilon_l \rangle}_{\in \mathbb{Z}}$

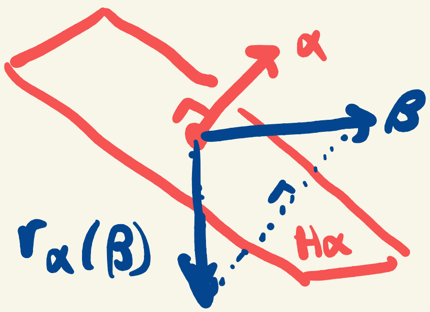
Properties of root space decomp of L motivate the axiomatic definition of a <u>root system</u> (of which Φ is an example)

Let E be a finite dim real vector space ($\cong \mathbb{R}^n$, some n) with a <u>symmetric</u>, <u>positive definite</u> bilinear form $\underbrace{(\cdot,\cdot)}$ → e.g. standard inner product on $\mathbb{R}^n$

$(\alpha,\beta) = (\beta,\alpha)$ $\quad$ $(\alpha,\alpha) > 0$ if $0 \neq \alpha \in E$

[Recall that $\|\alpha\| = \sqrt{(\alpha,\alpha)}$ and $(\alpha,\beta) = \|\alpha\|\,\|\beta\| \cos\theta$
where θ is angle between α and β.

For $0 \neq \alpha \in E$ define $r_\alpha : E \to E$ by $r_\alpha(\beta) \overset{\text{def}}{=}$ (the vector obtained by reflecting β across the hyperplane $H_\alpha = \{v \in E \mid (\alpha, v) = 0\} = (\mathbb{R}\alpha)^\perp$)



If $c \in \mathbb{R}$ is such that $\beta - c\alpha \in H_\alpha$, then $r_\alpha(\beta) = \beta - 2c\alpha$. But $\beta - c\alpha \in H_\alpha \Rightarrow (\beta - c\alpha, \alpha) = 0$
$\Rightarrow (\beta,\alpha) = c(\alpha,\alpha) \Rightarrow c = (\beta,\alpha)/(\alpha,\alpha)$.

general linear group

Thus the reflection $r_\alpha : E \to E$ belongs to $GL(E)$

and has formula $\boxed{r_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha \quad \text{where } \langle \beta, \alpha \rangle \overset{\text{def}}{=} \frac{2(\beta,\alpha)}{(\alpha,\alpha)}}$

Thus: $\begin{cases} r_\alpha^{-1} = r_\alpha \\ r_{c\alpha} = r_\alpha \text{ if } 0 \neq c \in \mathbb{R} \\ (r_\alpha(\beta), r_\alpha(\gamma)) = (\beta, \gamma) \end{cases}$

Def A subset $\Phi \subseteq E$ is root system if

(R1) $|\Phi| < \infty$ and $0 \notin \Phi$ and Φ spans E

(R2) If $\alpha \in \Phi$ then $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$

(R3) If $\alpha \in \Phi$ then $r_\alpha(\beta) \in \Phi$ $\forall \beta \in \Phi$

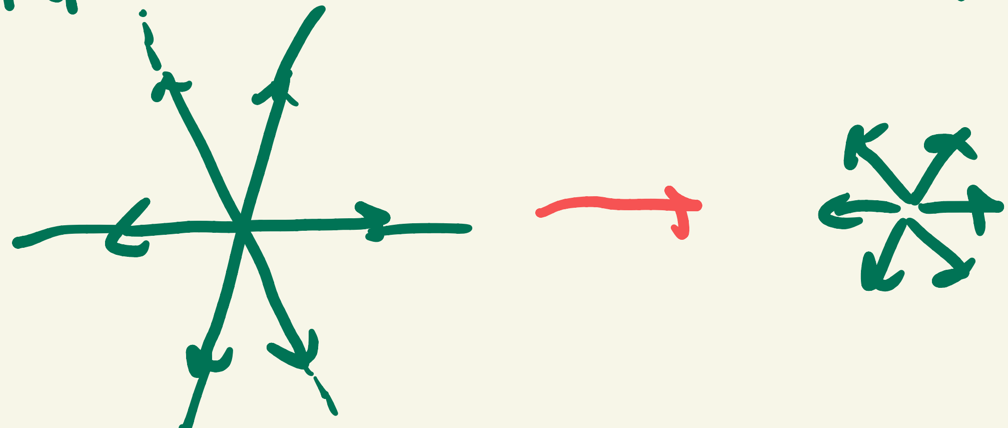
(R4) If $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle \in \mathbb{Z}$

The Weyl group ("vial") of Φ is $W = \langle r_\alpha \mid \alpha \in \Phi \rangle \subseteq GL(E)$

Since Φ is finite, and spans E, and since each r_α defines a permutation of Φ, it follows that W is isomorphic to a subgroup of the symmetric group of all permutations of Φ. Thus the Weyl group has $|W| < \infty$.

Quick intuitive idea for root system:

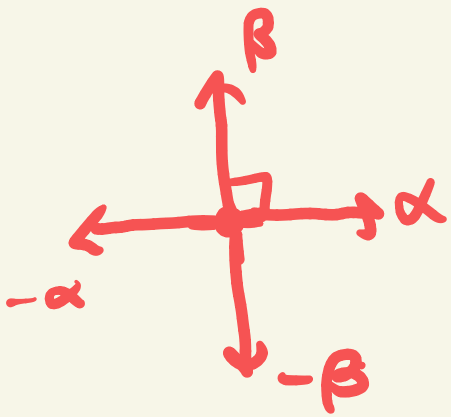
Suppose W is any finite subgroup of E generated by reflections r_α.
Consider the set lines $\mathbb{R}\alpha$ for $\alpha \neq 0$ with $r_\alpha \in W$.
Replace each of these lines by a pair of vectors α and $-\alpha$



(Morally) the result is a root system with Weyl group W, and any root system arises like this

Examples of root systems when $E = \mathbb{R}^2$ with standard inner product

$\Phi_{A_1 \times A_1}$:



4 roots, $(\alpha, \beta) = \langle \alpha, \beta \rangle = 0$

$r_\alpha : \alpha \mapsto -\alpha$
$-\alpha \mapsto \alpha$
$\beta \mapsto \beta$
$-\beta \mapsto -\beta$

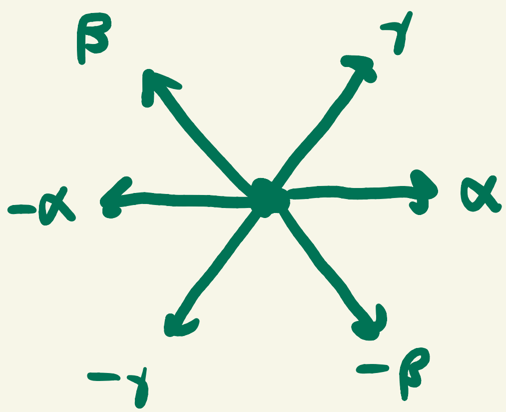
r_β fixes $\pm\alpha$
negates $\pm\beta$

$\Rightarrow W = \langle r_\alpha, r_\beta \rangle$
$\cong S_2 \times S_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

In this example, α and β could have different lengths

$r_\alpha r_\beta = r_\beta r_\alpha$

Φ_{A_2}:



6 roots, diagonals of a regular hexagon

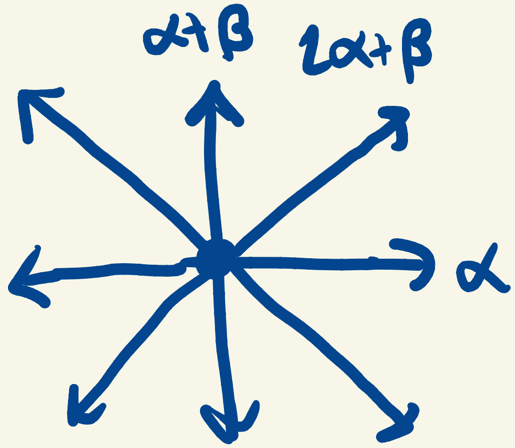
$$(\alpha, \beta) = \|\alpha\| \|\beta\| \cos \frac{2\pi}{3} = \frac{-\|\beta\|^2}{2}$$

$$\Rightarrow \langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\beta, \beta)} = -1$$, similarly for $\langle \alpha, \gamma \rangle$, $\langle \beta, \gamma \rangle$

$r_\alpha : \begin{bmatrix} \beta \leftrightarrow \gamma & \alpha \leftrightarrow -\alpha \\ -\beta \leftrightarrow -\gamma & \end{bmatrix}$ $r_\beta : \begin{bmatrix} \alpha \leftrightarrow \gamma & \beta \leftrightarrow -\beta \\ -\alpha \leftrightarrow -\gamma & \end{bmatrix}$ $r_\gamma : \begin{bmatrix} \alpha \leftrightarrow -\beta & \gamma \leftrightarrow -\gamma \\ \beta \leftrightarrow -\alpha & \end{bmatrix}$

Can check that $W = \langle r_\alpha, r_\beta, r_\gamma \rangle \cong S_3$

Φ_{B_2}:



8 roots, $\|\beta\| = \sqrt{2}\,\|\alpha\|$

$\|\alpha+\beta\| = \|\alpha\|$

$$\langle \alpha, \beta \rangle = \frac{2\|\alpha\|\|\beta\|\cos\left(\frac{3\pi}{2}\right)}{\|\beta\|^2} = \sqrt{2}\cdot\frac{-1}{\sqrt{2}} = -1$$

likewise with other inner products $\langle \cdot, \cdot \rangle$

$r_\alpha : \pm\beta \leftrightarrow \pm(2\alpha+\beta)$

$\alpha \leftrightarrow -\alpha$

$\pm(\alpha+\beta)$ is fixed

$r_\beta : \pm\alpha \leftrightarrow \pm(\alpha+\beta)$

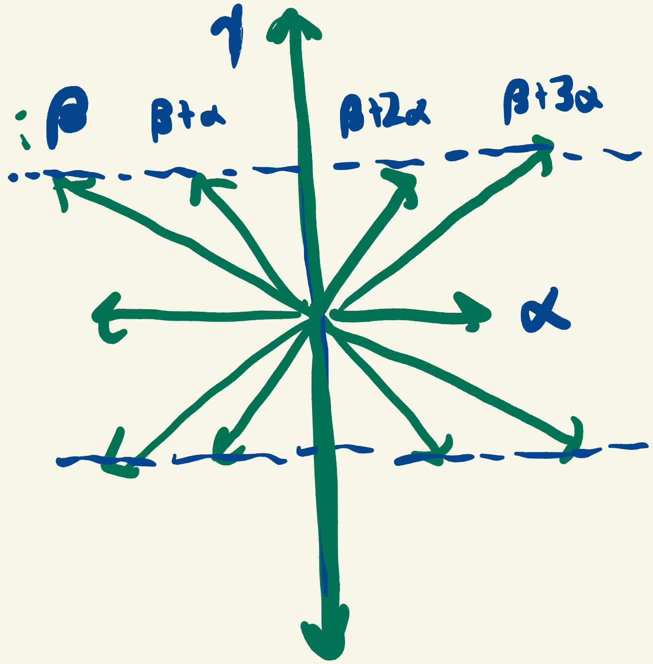
$\beta \leftrightarrow -\beta$

$\pm(2\alpha+\beta)$ fixed

Can show that $W = \langle r_\alpha, r_\beta \rangle \cong \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle$

$r_\alpha r_\beta r_\alpha r_\beta = id$ and $|W| = 8$

Φ_{G_2}



12 roots, a set of 6 short roots that look like Φ_{A_2} and a set of 6 long roots that also look like Φ_{A_2}

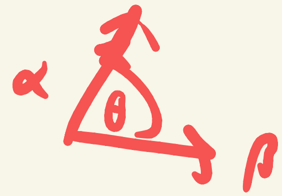
Can show that $W \cong$ dihedral group of order 12

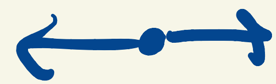
[Note: in these examples, always have $|W| = |\Phi|$]

Pairs of roots

The rank of Φ is $\dim E$.

Examples above are rank 2. (Only Φ of rank 1 is




)

Suppose $\alpha, \beta \in \Phi$ and $\beta \neq \pm\alpha$. Then

$$\langle \beta, \alpha \rangle = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} = 2\frac{\|\beta\|}{\|\alpha\|}\cos\theta \in \mathbb{Z}$$

$$\langle \beta, \alpha \rangle \langle \alpha, \beta \rangle = 4\cos^2\theta \in \mathbb{Z}$$

as $\cos^2\theta \in [0,1]$, the only possibilities for $\langle \alpha, \beta \rangle$, $\langle \beta, \alpha \rangle$, θ, $\|\beta\|^2/\|\alpha\|^2$, Φ are as follows:

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\lVert\beta\rVert^2 / \lVert\alpha\rVert^2$	Φ
0	0	$\pi/2$	?	$A_1 \times A_1$
1	1	$\pi/3$	1	A_2
-1	-1	$2\pi/3$	1	A_2
1	2	$\pi/4$	2	B_2
-1	-2	$3\pi/4$	2	B_2
1	3	$\pi/6$	3	G_2
-1	-3	$5\pi/6$	3	G_2

So in fact the four examples given account for all possible rank two root systems (up to isomorphism)