Math 5143 - Lecture 14

Math 5143 - Lecture 14

Last time: (abstract) root systems

Fix a finite dim. real vector space E with a bilinear form $(\cdot,\cdot)$ that is symmetric, positive definite

[By appropriately choosing bases, can identify E with $\mathbb{R}^n$ with standard inner product, but this may be inconvenient]

For $0 \neq \alpha \in E$, let $H_\alpha = \{v \in E \mid (v,\alpha) = 0\}$.

Then the reflection across H_α is the linear map

$$\boxed{r_\alpha : E \to E \quad \text{with formula} \quad r_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha}$$

Def A finite subset $\Phi \subseteq E \setminus \{0\}$ is a root system if

(R1) E is spanned by Φ

(R2) $\mathbb{R}\alpha \cap E = \{\pm\alpha\}$ for $\alpha \in \Phi$

(R3) $r_\alpha(\Phi) = \Phi \;\; \forall \alpha \in \Phi$

(R4) $\dfrac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z} \;\; \forall \alpha, \beta \in \Phi$

The elems of Φ are called roots

The subgroup of $GL(E)$ generated by $\{r_\alpha \mid \alpha \in \Phi\}$ is called the Weyl group of Φ, often denoted W

Notation: Set $\langle \beta, \alpha \rangle = \dfrac{2(\beta,\alpha)}{(\alpha,\alpha)}$ for $\alpha, \beta \in \Phi$

If $\Phi \subseteq E$ and $\Phi' \subseteq E'$ are root systems, then an isomorphism $\Phi \to \Phi'$ is a linear bijection $f : E \to E'$ such that $\langle f(\beta), f(\alpha) \rangle = \langle \beta, \alpha \rangle \; \forall \alpha, \beta \in \Phi$.

and $f(\Phi) = \Phi'$

<u>Motivation</u>: Suppose L is a semisimple Lie algebra, over $\mathbb{C}$, finite dim and nonzero. Choose a maximal <u>toral</u> subalgebra $H \subseteq L$ and let $H^* = \{\text{linear maps } H \to \mathbb{C}\}$.

↳ (all elements are semisimple)

(if L is classical, can take H to be subalgebra of diagonal matrices in L)

For each $\alpha \in H^*$ define $L_\alpha = \{x \in L \mid [h, x] = \alpha(h)x \ \forall h \in H\}$.

Set $\Phi = \{\alpha \in H^* \setminus 0 \mid L_\alpha \neq 0\}$. We showed $H = L_0$ is abelian.

So we have a decomposition $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$

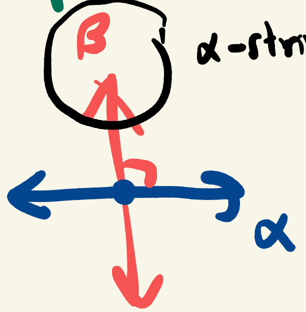
Here, $\boxed{\Phi \text{ is a root system}}$ in $E = \mathbb{R}\text{-span}\{\alpha \in \Phi\}$, where the relevant form $(\cdot, \cdot)$ is the Killing form of L, restricted to H, and then transferred to H^* by nondegeneracy.

Also: $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta} \quad \forall \alpha, \beta \in \Phi$

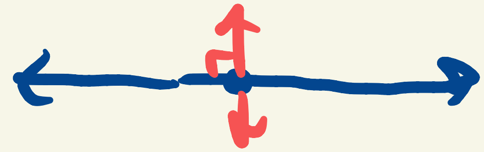
Up to isomorphism, there are 4 root systems in $\mathbb{R}^2$:

α-string has size 1

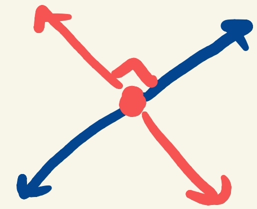
$\Phi_{A_1\times A_1}$



$\cong$

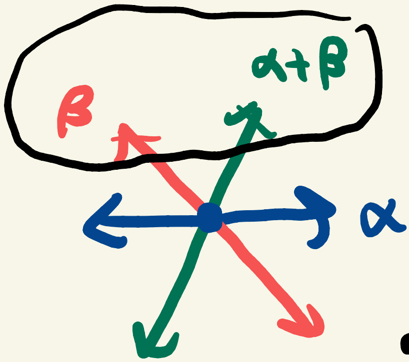


$\cong$



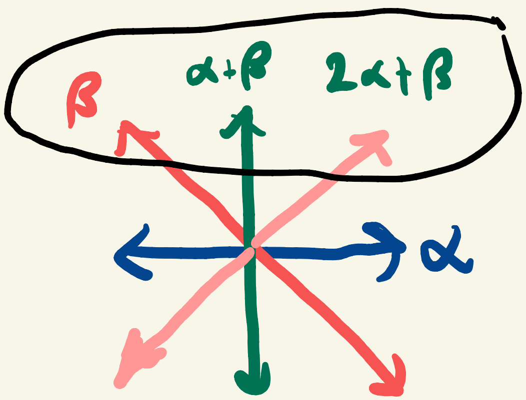
α-string has size 2

Φ_{A_2}



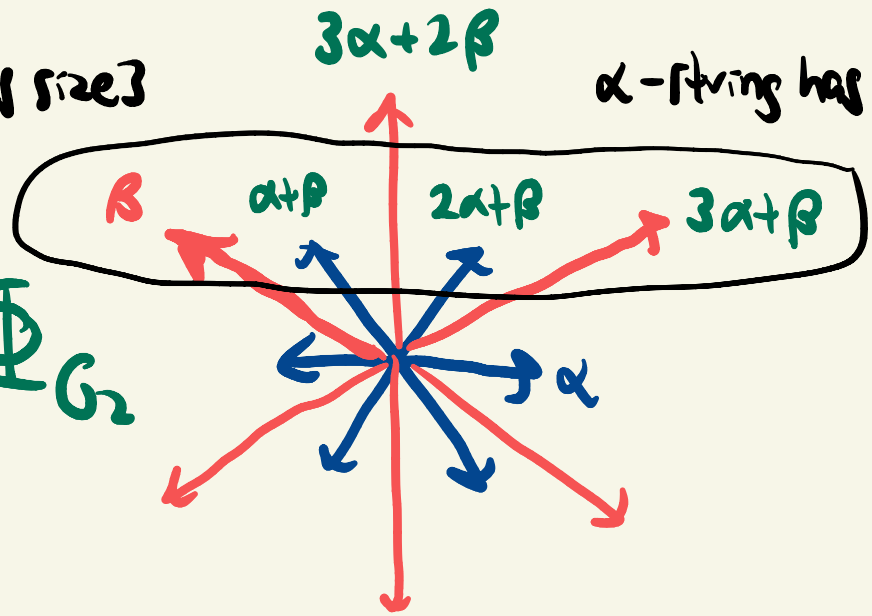
α-string has size 3

Φ_{B_2}



α-string has size 4

Φ_{G_2}



Prop. Let Φ be a root system with Weyl group W.
If $\sigma \in GL(E)$ has $\sigma(\Phi) = \Phi$ then $\sigma r_\alpha \sigma^{-1} = r_{\sigma(\alpha)}$ and $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ $\forall \alpha, \beta \in \Phi$.

Pf Compute $\sigma r_\alpha \sigma^{-1}(\sigma(\beta)) = \sigma r_\alpha(\beta) = \sigma(\beta) - \langle \beta, \alpha \rangle \sigma(\alpha)$.
Clearly $\sigma r_\alpha \sigma^{-1}$ preserves Φ and sends $\sigma(\alpha) \mapsto -\sigma(\alpha)$.
Also $\sigma r_\alpha \sigma^{-1}$ fixes the hyperplane $\sigma(H_\alpha)$ where $H_\alpha = \{v \in E \mid (v, \alpha) = 0\}$
A priori, we don't know that $\sigma(H_\alpha) = H_{\sigma(\alpha)}$. If we knew this then it would be clear by comparing formulas that $\sigma r_\alpha \sigma^{-1} = r_{\sigma(\alpha)}$ and also $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ $\forall \alpha, \beta \in \Phi$. So just need to show:

Lemma If $\sigma \in GL(E)$ has $\sigma(\Phi) = \Phi$ and σ fixes a hyperplane $H \subseteq E$ while sending some $0 \neq \alpha \in E$ to $-\alpha$, then $H = H_\alpha$ and $\sigma = \sigma_\alpha$.

this element must have $\alpha \notin H$

Pf idea (compare with text book)

Define $\tau = \sigma r_\alpha$. Then $\tau(\alpha) = \alpha$, $\tau(\Phi) = \Phi$, τ fixes H pt-wise

Choose a basis $v_1, v_2, \dots, v_{n-1}$ for H. Set $v_n = \alpha$.

Since $\alpha \notin H$, $v_1, v_2, \dots, v_n$ is a basis for E. But the matrix of τ in this basis is the identity matrix, so $\tau = 1$. $\square$

$\square$

Lemma Let $\alpha, \beta \in \Phi$ be non-proportional (so $\alpha \neq \pm\beta$)

(a) If $(\alpha, \beta) > 0$ then $\alpha - \beta \in \Phi$ (b) If $(\alpha, \beta) < 0$ then $\alpha + \beta \in \Phi$.

Pf (b) follows from (a), swapping β and $-\beta$. For (a): $(\alpha, \beta) > 0 \Rightarrow \langle \alpha, \beta \rangle > 0$.

The acute angle between α and β must be $\pi/3$, $\pi/4$, or $\pi/6$ (by considering the 4 root systems in $\mathbb{R}^2$) (since α, β not orthogonal) and must have $\langle \alpha, \beta \rangle = 1$ or $\langle \beta, \alpha \rangle = 1$.

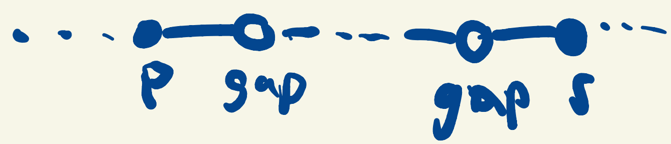
If $\langle \alpha, \beta \rangle = 1$ then $\alpha - \beta = \sigma_\beta(\alpha) \in \Phi$. If $\langle \beta, \alpha \rangle = 1$ then $\alpha - \beta = -\sigma_\alpha(\beta) \in \Phi$. $\square$

For $\alpha, \beta \in \Phi$, with $\beta \neq \pm\alpha$, the $\underline{\alpha\text{-string through } \beta}$ is the set of roots $\{\beta + i\alpha \mid i \in \mathbb{Z}\} \cap \Phi$.

↳ this sequence is finite but has no "gaps"

Prop. There are integers $q, r \geq 0$ such that the α-string through β is exactly $\{\beta + i\alpha \mid -r \leq i \leq q\}$.

Pf If there were any gaps in the string, then we could find $p, s \in \mathbb{Z}$ with $-r \leq p < s \leq q$ where $\beta + p\alpha, \beta + s\alpha \in \Phi$ but $\beta + (p+1)\alpha, \beta + (s-1)\alpha \notin \Phi$.



Prev lemma implies $(\beta + p\alpha, \alpha) \geq 0 \geq (\beta + s\alpha, \alpha)$

$\Rightarrow ((s-p)\alpha, \alpha) = |s-p|(\alpha, \alpha) \leq 0$, impossible as $(\cdot,\cdot)$ is pos. definite $\square$

__Cor__. The integers $r, q \geq 0$ such that the α-string through β is $\{\beta + i\alpha \mid -r \leq i \leq q\}$ satisfy $r - q = \langle \beta, \alpha \rangle \in \{0, \pm 1, \pm 2, \pm 3\}$

So every α-string has at most 4 elements.

__Pf__. The reflection r_α preserve (and in fact, reverses) the α-string through β since $r_\alpha(\beta + i\alpha) = \beta - (\underbrace{\langle \beta, \alpha \rangle}_{\in \mathbb{Z}} + i)\alpha$. Therefore must have $r_\alpha(\beta + q\alpha) = \beta - r\alpha$. But $r_\alpha(\beta + q\alpha) = \beta - \langle \beta, \alpha \rangle \alpha - q\alpha$ so $\langle \beta, \alpha \rangle = r - q$. $\square$