SOLUTIONS TO FINAL EXAMINATION – MATH 2121, FALL 2024

Problem 1. (20 points) In the following statements, A, B, C, etc., are matrices (with all real entries), and $u, v, w, x,$, etc., are vectors in $\mathbb{R}^n$, unless otherwise noted.

(1) Any system of n linear equations in n variables has at least n solutions.

TRUE **[FALSE]**

(2) If a linear system $Ax = b$ has more than one solution, then so does $Ax = 0$.

[TRUE] FALSE

(3) If A and B are $n \times n$ matrices with $AB = 0$, then $A = 0$ or $B = 0$.

TRUE **[FALSE]**

(4) If $AB = BA$ and A is invertible, then $A^{-1}B = BA^{-1}$.

[TRUE] FALSE

(5) If A is a square matrix, then $\det(-A) = -\det A$.

TRUE **[FALSE]**

(6) If A is a nonzero matrix then $\det(A^\top A) > 0$.

TRUE **[FALSE]**

(7) If A is the $m \times n$ standard matrix of a one-to-one linear transformation, then $\mathrm{rank}(A) = m$.

TRUE **[FALSE]**

(8) If V is a vector space and $S \subset V$ is a subset whose span is V, then some subset of S is a basis of V.

[TRUE] FALSE

(9) If A is square and contains a row of zeros, then 0 is an eigenvalue of A.

[TRUE] FALSE

(10) Each eigenvector of a square matrix A is also an eigenvector of A^2.

[TRUE] FALSE

(11) If A is diagonalizable, then the columns of A are linearly independent.

TRUE **[FALSE]**

(12) Every 2×2 matrix (with all real entries) has an eigenvector in $\mathbb{R}^2$.

TRUE **[FALSE]**

(13) Every 3×3 matrix (with all real entries) has an eigenvector in $\mathbb{R}^3$.

[TRUE] FALSE

(14) If the entries in each column of a square matrix A sum to 1, then $\lambda = 1$ is an eigenvalue of A.

[TRUE] FALSE

(15) If the columns of A are orthonormal then $A^\top A$ is an identity matrix.

[TRUE] FALSE

(16) If A is a 2×2 matrix such that Av is always orthogonal to $v \in \mathbb{R}^2$, then A cannot be invertible.

TRUE **[FALSE]**

(17) If A is a 3×3 matrix such that Av is always orthogonal to $v \in \mathbb{R}^3$, then A cannot be invertible.

[TRUE] FALSE

(18) If A is an $m \times n$ matrix and the linear system $Ax = b$ has more basic variables than free variables, then $\operatorname{rank} A > \frac{n}{2}$.

[TRUE] FALSE

(19) If A is an $n \times n$ matrix with a negative real eigenvalue, then A is not symmetric.

TRUE **[FALSE]**

(20) Any polynomial of the form $a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} + x^n$ with $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$ can occur as $\det(xI - A)$ for some $n \times n$ matrix A.

[TRUE] FALSE

Problem 2. (10 points = 1 + 1 + 1 + 1 + 1 + 1 + 2 + 2)

Suppose $a, b \in \mathbb{R}$ are NONZERO real numbers. Define the matrix $A = \begin{bmatrix} a \\ b \end{bmatrix}$.

This question has 8 parts.

Your answers to some parts may depend on the values of a and b.

(1) The formula $T(x) = Ax$ defines a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ for which values of m and n?

$\boxed{m = 2}$ $\boxed{n = 1}$

(2) Compute the reduced echelon form of $A = \begin{bmatrix} a \\ b \end{bmatrix}$.

$\boxed{\mathsf{RREF}(A) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$

(3) For which values of a and b is $T : \mathbb{R}^n \to \mathbb{R}^m$ from part (1) one-to-one?

$\boxed{\text{all values of } a \text{ and } b}$

(4) For which values of a and b is $T : \mathbb{R}^n \to \mathbb{R}^m$ from part (1) onto?

$\boxed{\text{no values}}$

(5) Find a basis for the column space of $A = \begin{bmatrix} a \\ b \end{bmatrix}$.

$\boxed{\begin{bmatrix} a \\ b \end{bmatrix}}$

(6) Find a basis for the null space of $A = \begin{bmatrix} a \\ b \end{bmatrix}$.

$\boxed{\text{the empty set}}$ since $\mathsf{Nul}(A) = \{0\}$.

(7) Let $y \in \mathbb{R}^m$ be an arbitrary vector.

Compute all solutions $x \in \mathbb{R}^n$ to $Ax = y$ when $A = \begin{bmatrix} a \\ b \end{bmatrix}$.

Solution:

Suppose $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. The unique solution to $Ax = y$ is $\boxed{x = \frac{y_1}{a} = \frac{y_2}{b}}$ if the two fractions are equal, and otherwise $Ax = y$ has $\boxed{\text{no solution}}$.

(8) Find a singular value decomposition for $A = \begin{bmatrix} a \\ b \end{bmatrix}$.

Solution:

Since $A^\top A = \begin{bmatrix} a^2 + b^2 \end{bmatrix}$, the unique singular value of A is $\sigma_1 = \sqrt{a^2+b^2}$.

So the SVD of A will have the form $A = U\Sigma V^\top$ where

$$\boxed{\Sigma = \begin{bmatrix} \sqrt{a^2+b^2} \\ 0 \end{bmatrix}}$$

and where U is orthogonal and 2×2 and V is an orthogonal and 1×1.

The only choices for V are $\begin{bmatrix} \pm 1 \end{bmatrix}$ so let us just set

$$\boxed{V = \begin{bmatrix} 1 \end{bmatrix}}.$$

We need an orthogonal 2×2 matrix U such that $U \begin{bmatrix} \sqrt{a^2+b^2} \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$.

The first column of U must be $\frac{1}{\sqrt{a^2+b^2}} \begin{bmatrix} a \\ b \end{bmatrix}$ and then the second column, which must be a unit vector orthogonal to the first column, is either

$$\frac{1}{\sqrt{a^2+b^2}} \begin{bmatrix} -b \\ a \end{bmatrix} \quad \text{or} \quad \frac{1}{\sqrt{a^2+b^2}} \begin{bmatrix} b \\ -a \end{bmatrix}.$$

So we must have

$$U = \tfrac{1}{\sqrt{a^2+b^2}} \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \quad \text{or} \quad \tfrac{1}{\sqrt{a^2+b^2}} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Our final SVD is

$$\boxed{\begin{bmatrix} a \\ b \end{bmatrix} = \tfrac{1}{\sqrt{a^2+b^2}} \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} \sqrt{a^2+b^2} \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}^\top}.$$

Problem 3. (10 points = 1 + 1 + 1 + 1 + 1 + 1 + 2 + 2)

Suppose $p, q \in \mathbb{R}$ are NONZERO real numbers. Define the matrix $B = \begin{bmatrix} p & q \end{bmatrix}$.

This question has 8 parts.

Your answers to some parts may depend on the values of p and q.

(1) The formula $U(x) = Bx$ defines a linear transformation $U : \mathbb{R}^n \to \mathbb{R}^m$ for which values of m and n?

$\boxed{m = 1}$ $\boxed{n = 2}$

(2) Compute the reduced echelon form of $B = \begin{bmatrix} p & q \end{bmatrix}$.

$\boxed{\mathsf{RREF}(B) = \begin{bmatrix} 1 & q/p \end{bmatrix}}$

(3) For which values of p and q is $U : \mathbb{R}^n \to \mathbb{R}^m$ from part (1) one-to-one?

$\boxed{\text{no values}}$

(4) For which values of p and q is $U : \mathbb{R}^n \to \mathbb{R}^m$ from part (1) onto?

$\boxed{\text{all values of } p \text{ and } q}$ since these are both nonzero.

(5) Find a basis for the column space of $B = \begin{bmatrix} p & q \end{bmatrix}$.

$\boxed{\begin{bmatrix} 1 \end{bmatrix}}$

(6) Find a basis for the null space of $B = \begin{bmatrix} p & q \end{bmatrix}$.

$$\boxed{\begin{bmatrix} -q \\ p \end{bmatrix}}$$

(7) Let $y \in \mathbb{R}^m$ be an arbitrary vector.

Compute all solutions $x \in \mathbb{R}^n$ to $Bx = y$ when $B = \begin{bmatrix} p & q \end{bmatrix}$.

Solution:

As $y \in \mathbb{R}^1 = \mathbb{R}$ the solutions are $\boxed{x = \begin{bmatrix} y/p \\ 0 \end{bmatrix} + t \begin{bmatrix} -q \\ p \end{bmatrix}}$ for any $t \in \mathbb{R}$.

(8) Find a singular value decomposition for $B = \begin{bmatrix} p & q \end{bmatrix}$.

Solution:

We have $B^\top B = \begin{bmatrix} p^2 & pq \\ pq & q^2 \end{bmatrix}$ so

$$\det(B^\top B - xI) = (p^2 - x)(q^2 - x) - p^2 q^2 = x^2 - (p^2 + q^2)x.$$

This means the eigenvalues of $B^\top B$ are $p^2 + q^2$ and 0.

So the singular values of B are $\sqrt{p^2 + q^2}$ and 0.

Thus the SVD of B will have the form $B = U\Sigma V^\top$ where

$$\boxed{\Sigma = \begin{bmatrix} \sqrt{p^2 + p^2} & 0 \end{bmatrix}}$$

and where U is orthogonal and 1×1 and V is an orthogonal and 2×2.

The only choices for U are $\begin{bmatrix} \pm 1 \end{bmatrix}$ so let us just set

$$\boxed{U = \begin{bmatrix} 1 \end{bmatrix}}.$$

We need an orthogonal 2×2 matrix $V^\top$ such that

$$\begin{bmatrix} \sqrt{p^2 + q^2} & 0 \end{bmatrix} V^\top = \begin{bmatrix} p & q \end{bmatrix}.$$

The first row of $V^\top$ must be $\frac{1}{\sqrt{p^2+q^2}} \begin{bmatrix} p & q \end{bmatrix}$ and then the second row, which must be a unit vector orthogonal to the first row, is either

$$\frac{1}{\sqrt{p^2 + p^2}} \begin{bmatrix} -q & p \end{bmatrix} \quad \text{or} \quad \frac{1}{\sqrt{p^2 + q^2}} \begin{bmatrix} q & -p \end{bmatrix}.$$

So we must have

$$V^\top = \frac{1}{\sqrt{p^2+q^2}} \begin{bmatrix} p & q \\ -q & p \end{bmatrix} \quad \text{or} \quad \frac{1}{\sqrt{p^2+q^2}} \begin{bmatrix} p & q \\ q & -p \end{bmatrix}.$$

Our final SVD is

$$\boxed{\begin{bmatrix} p & q \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} \sqrt{p^2 + q^2} & 0 \end{bmatrix} \frac{1}{\sqrt{p^2+q^2}} \begin{bmatrix} p & -q \\ q & p \end{bmatrix}^\top}.$$

Problem 4. (20 points = 1 + 1 + 1 + 1 + 4 + 4 + 4 + 4)

Suppose $a, b, p, q \in \mathbb{R}$ are NONZERO real numbers.

Again define the matrices $A = \begin{bmatrix} a \\ b \end{bmatrix}$ and $B = \begin{bmatrix} p & q \end{bmatrix}$. Then let $M = AB$.

This question has 8 parts.

Your answers to some parts may depend on the values of a, b, p, and q.

(1) Compute the matrix M.

$$\boxed{M = \begin{bmatrix} ap & aq \\ bp & bq \end{bmatrix}}$$

(2) Compute the reduced echelon form of M.

$$M = \begin{bmatrix} ap & aq \\ bp & bq \end{bmatrix} \to \begin{bmatrix} p & q \\ p & q \end{bmatrix} \to \begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix} \to \boxed{\mathsf{RREF}(M) = \begin{bmatrix} 1 & q/p \\ 0 & 0 \end{bmatrix}}.$$

(3) Find a basis for the column space of M.

$\boxed{\begin{bmatrix} a \\ b \end{bmatrix}}$ is a basis. $\boxed{\begin{bmatrix} ap \\ bp \end{bmatrix}}$ also works.

(4) Find a basis for the null space of M.

$\boxed{\begin{bmatrix} q \\ -p \end{bmatrix}}$ is a basis. $\boxed{\begin{bmatrix} -q/p \\ 1 \end{bmatrix}}$ also works.

(5) Find the eigenvalues of M.

Solution:

We have

$$\det(M - xI) = (ap - x)(bq - x) - abpq = x^2 - (ap + bq)x$$

so the eigenvalues are $\boxed{\lambda_1 = ap + bq \quad \text{and} \quad \lambda_2 = 0}$.

(6) For each eigenvalue of M, find a basis for the corresponding eigenspace.

Solution:

Since $M - \lambda_1 I = \begin{bmatrix} ap & aq \\ bp & bq \end{bmatrix} - \begin{bmatrix} ap+bq & 0 \\ 0 & ap+bq \end{bmatrix} = \begin{bmatrix} -bq & aq \\ bp & -ap \end{bmatrix}$ has
$\mathsf{RREF}(\begin{bmatrix} -bq & aq \\ bp & -ap \end{bmatrix}) = \mathsf{RREF}(\begin{bmatrix} -b & a \\ b & -a \end{bmatrix}) = \begin{bmatrix} 1 & -a/b \\ 0 & 0 \end{bmatrix}$
a basis the $(ap + bq)$-eigenspace is $\boxed{\begin{bmatrix} a \\ b \end{bmatrix}}$.

A basis the 0-eigenspace is $\boxed{\begin{bmatrix} q \\ -p \end{bmatrix}}$.

(7) Determine when is M diagonalizable.

In the case when M is diagonalizable, find an invertible matrix P and a diagonal matrix D such that $M = PDP^{-1}$.

Solution:

M is diagonalizable iff $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} q \\ -p \end{bmatrix}$ are linearly independent.

This occurs iff $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} p \\ q \end{bmatrix}$ are NOT orthogonal, meaning $\boxed{ap + bq \neq 0}$.

In this case $M = PDP^{-1}$ for $\boxed{P = \begin{bmatrix} a & q \\ b & -p \end{bmatrix}}$ and $\boxed{D = \begin{bmatrix} ap+bq & 0 \\ 0 & 0 \end{bmatrix}}$.

(8) Let $y \in \mathbb{R}^2$ be an arbitrary vector.

Solution:

Write $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.

If $\frac{y_1}{a} = \frac{y_2}{b}$ then the solutions are $\boxed{x = \begin{bmatrix} \frac{y_1}{ap} \\ 0 \end{bmatrix} + t \begin{bmatrix} q \\ -p \end{bmatrix}}$ for any $t \in \mathbb{R}$.

Otherwise there are $\boxed{\text{no solutions}}$.

Problem 5. (10 points) Consider the matrix

$$A = \begin{bmatrix} 0 & a & b & 0 \\ c & 0 & 0 & d \\ e & 0 & 0 & f \\ 0 & g & h & 0 \end{bmatrix}$$

where $a, b, c, d, e, f, g, h \in \mathbb{R}$.

Find formulas for $\det(A)$ and A^{-1} (assuming the inverse exists).

Solution:

We can do 2 column swaps to change A to $\begin{bmatrix} a & b & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & e & f \\ g & h & 0 & 0 \end{bmatrix}$. Then we can do 2 row swaps to change this matrix to

$$B = \begin{bmatrix} a & b & 0 & 0 \\ g & h & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & e & f \end{bmatrix}.$$

This matrix is block diagonal with determinant

$$\det(B) = \boxed{\det(A) = (ah - bg)(cf - de)}$$

since $(-1)^4 = 1$. Notice that

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The outer matrices are invertible, so

$$A^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} B^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The inverse of B is

$$\begin{bmatrix} \frac{h}{ah-bg} & \frac{-b}{ah-bg} & 0 & 0 \\ \frac{-g}{ah-bg} & \frac{a}{ah-bg} & 0 & 0 \\ 0 & 0 & \frac{f}{cf-de} & \frac{-d}{cf-de} \\ 0 & 0 & \frac{-e}{cf-de} & \frac{c}{cf-de} \end{bmatrix}.$$

Thus, when invertible, A has inverse

$$\boxed{A^{-1} = \begin{bmatrix} 0 & \frac{f}{cf-de} & \frac{-d}{cf-de} & 0 \\ \frac{h}{ah-bg} & 0 & 0 & \frac{-b}{ah-bg} \\ \frac{-g}{ah-bg} & 0 & 0 & \frac{a}{ah-bg} \\ 0 & \frac{-e}{cf-de} & \frac{c}{cf-de} & 0 \end{bmatrix}}.$$

Problem 6. (10 points) Suppose we have 4 datapoints

$$(x_1, y_1) = (-1, 3), \quad (x_2, y_2) = (0, 1), \quad (x_3, y_3) = (1, 0), \quad (x_4, y_4) = (2, 8)$$

The **parabola of best fit** for these datapoints is the function

$$f(x) = a + bx + cx^2 \qquad \text{for some real parameters } a, b, c$$

that minimizes the error defined by

$$|f(x_1) - y_1|^2 + |f(x_2) - y_2|^2 + |f(x_3) - y_3|^2 + |f(x_4) - y_4|^2.$$

Find an exact formula for a vector of parameters

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

that define a parabolic of best fit for the given data.

Your formula does not need to be completely simplified: it is sufficient to give the answer as a product of one or more numeric matrices or their inverses—as long as your answer is a numeric expression that we could enter into a matrix calculator.

Solution:

The parameters will be a least-squares solution to $A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = y$ for

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 8 \end{bmatrix}.$$

The columns of A are linearly independent so $A^\top A$ is invertible. Thus the unique least-squares solution is $(A^\top A)^{-1} A^\top y$:

$$\boxed{\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \\ 8 \end{bmatrix}}.$$

This simplifies to $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -0.2 \\ -1.1 \\ 2.5 \end{bmatrix}$, though you did not need to compute this.

Problem 7. (10 points = 5 + 5) Suppose $a, b, c \in \mathbb{R}$ are real numbers with $a \neq 0$.

This problem has two parts.

(a) Find an orthogonal basis for the subspace

$$V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : ax_1 + bx_2 + cx_3 = 0. \right\}$$

such that **all entries of your basis vectors are polynomials in** a**,** b**, and** c.

This means that the entries in your basis vectors should not involve any square roots or fractions with a, b, or c in the denominator.

Solution:

$V = \mathsf{Nul}(\begin{bmatrix} a & b & c \end{bmatrix})$ and $\mathsf{RREF}(\begin{bmatrix} a & b & c \end{bmatrix}) = \begin{bmatrix} 1 & b/a & c/a \end{bmatrix}$

Therefore V is 2-dimensional and one basis is $\begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}, \begin{bmatrix} -c \\ 0 \\ a \end{bmatrix}$.

This basis is not orthogonal. We could do the Gram-Schmidt process to convert the second vector to one that is orthogonal the first.

We can accomplish the same thing in a little less time by observing that any vector orthogonal to $\begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$ must be a scalar multiple of $\begin{bmatrix} a \\ b \\ z \end{bmatrix}$ where z can be arbitrary. Since we want this vector to be in V, we could just choose z such that $a^2 + b^2 + cz = 0$.

If $c \neq 0$ then $z = \frac{-a^2-b^2}{c}$ and our second basis vector is $\begin{bmatrix} a \\ b \\ \frac{-a^2-b^2}{c} \end{bmatrix}$. This vector does not have polynomial entries, but if we multiply it by c then we get a vector that works: $\begin{bmatrix} ac \\ bc \\ -a^2 - b^2 \end{bmatrix}$.

The same vector works when $c = 0$ so one answer is $\boxed{\begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}, \begin{bmatrix} ac \\ bc \\ -a^2 - b^2 \end{bmatrix}}$.

Other answers are possible, and it is easy to check if a given answer works by just computing dot products.

(b) Find a matrix Q such that $Qy = \mathrm{proj}_V(y)$ for all $y \in \mathbb{R}^3$.

As in problem 6, your answer for Q does not need to be completely simplified to receive full credit.

Solution:

Let $v = \begin{bmatrix} -b \\ a \\ 0 \end{bmatrix}$ and $w = \begin{bmatrix} ac \\ bc \\ -a^2 - b^2 \end{bmatrix}$. Then

$$\mathrm{proj}_V(y) = \frac{v \bullet y}{v \bullet v} v + \frac{w \bullet y}{w \bullet w} w.$$

Remember that

$$\frac{v \bullet y}{v \bullet v} = (v \bullet v)^{-1} v^\top y \quad \text{and} \quad \frac{w \bullet y}{w \bullet w} = (w \bullet w)^{-1} w^\top y.$$

So we can express

$$\mathrm{proj}_V(y) = \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} (v \bullet v)^{-1} v^\top \\ (w \bullet w)^{-1} w^\top \end{bmatrix} y = \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} v \bullet v & 0 \\ 0 & w \bullet w \end{bmatrix}^{-1} \begin{bmatrix} v^\top \\ w^\top \end{bmatrix} y$$

so our answer is

$$Q = \begin{bmatrix} v & w \end{bmatrix} \begin{bmatrix} v \bullet v & 0 \\ 0 & w \bullet w \end{bmatrix}^{-1} \begin{bmatrix} v^\top \\ w^\top \end{bmatrix}.$$

Let us simply this a little bit:

$$\boxed{Q = \begin{bmatrix} -b & ac \\ a & bc \\ 0 & -a^2 - b^2 \end{bmatrix} \begin{bmatrix} a^2 + b^2 & 0 \\ 0 & (a^2 + b^2)(a^2 + b^2 + c^2) \end{bmatrix}^{-1} \begin{bmatrix} -b & a & 0 \\ ac & bc & -a^2 - b^2 \end{bmatrix}}.$$

This product can be simplified more, but this expression is sufficient.

Problem 8. (10 points)

Define $V = \{M \in \mathbb{R}^{3\times 3} : AM = MA\}$ where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -10 & 3 \end{bmatrix}$.

Here $\mathbb{R}^{3\times 3}$ means the vector space of 3×3 matrices with all entries in $\mathbb{R}$.

Explain why V is a subspace. Then find a basis for V.

Solution:

V is a subspace since it contains the zero matrix as $A0 = 0A = 0$, and if $M, N \in V$ and $c \in \mathbb{R}$ then $M + N \in V$ and $cM \in V$ since

$$(M + N)A = MA + NA = AM + AN = A(M + N)$$

and $(cM)A = c(MA) = c(AM) = A(cM)$.

To find a basis for V, we can try to solve the linear system

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -10 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -10 & 3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix}$$

which simplifies to

$$\begin{bmatrix} 0 & x_1 - 10x_3 & x_2 + 3x_3 \\ 0 & x_4 - 10x_6 & x_5 + 3x_6 \\ 0 & x_7 - 10x_9 & x_8 + 3x_9 \end{bmatrix} = \begin{bmatrix} x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \\ -10x_4 + 3x_7 & -10x_5 + 3x_8 & -10x_6 + 3x_9 \end{bmatrix}.$$

Examining the first column tells us that $x_4 = x_7 = 0$. This leaves us with

$$\begin{aligned} x_1 - 10x_3 - x_5 &= 0 \\ -10x_6 - x_8 &= 0 \\ 10x_5 - 3x_8 - 10x_9 &= 0 \\ x_2 + 3x_3 - x_6 &= 0 \\ x_5 + 3x_6 - x_9 &= 0 \\ 10x_6 + x_8 &= 0 \end{aligned}$$

which we can write as

$$\begin{bmatrix} 1 & 0 & -10 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10 & -1 & 0 \\ 0 & 0 & 0 & 10 & 0 & -3 & -10 \\ 0 & 1 & 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 10 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_5 \\ x_6 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now we need to row reduce

$$A = \begin{bmatrix} 1 & 0 & -10 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -10 & -1 & 0 \\ 0 & 0 & 0 & 10 & 0 & -3 & -10 \\ 0 & 1 & 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 0 & 10 & 1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -10 & 0 & 0 & -.3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & .1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -.3 & -1 \\ 0 & 0 & 0 & 0 & 1 & .1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -10 & 0 & 0 & -.3 & -1 \\ 0 & 1 & 3 & 0 & 0 & .1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -.3 & -1 \\ 0 & 0 & 0 & 0 & 1 & .1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are 3 non-pivot columns so the subspace V is 3-dimensional.

The free variables are x_3, x_8, and x_9, so we can find a basis by setting $(x_3, x_8, x_9) = (1, 0, 0)$ or $(0, 1, 0)$ or $(0, 0, 1)$. This leads to the solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_5 \\ x_6 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} .3 \\ -.1 \\ 0 \\ .3 \\ -.1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

which correspond to the basis

$$\begin{bmatrix} 10 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} .3 & -.1 & 0 \\ 0 & .3 & -.1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

A nicer answer with all integer entries is the basis

$$\boxed{\begin{bmatrix} 10 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & 10 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

Problem 9. (10 points)

Find all symmetric 2×2 matrices A with all real entries such that

$$\mathsf{trace}(A) = 0 \quad \text{and} \quad \lim_{n\to\infty} A^{2n} \text{ exists.}$$

Compute the value of the limit for each of these matrices.

Justify your answer to receive full credit.

Solution:

Such a matrix must have the form $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ for some $a, b \in \mathbb{R}$. But then

$$A^2 = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \begin{bmatrix} a & b \\ b & -a \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{bmatrix}$$

so we have

$$A^{2n} = \begin{bmatrix} (a^2 + b^2)^n & 0 \\ 0 & (a^2 + b^2)^n \end{bmatrix}.$$

This has a limit as $n \to \infty$ if and only if $a^2 + b^2 \leq 1$. If $a^2 + b^2 = 1$ then the limit is the identity matrix and if $a^2 + b^2 < 1$ then the limit if the zero matrix.

So our answer is:

$$\boxed{A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix} \text{ with } a^2 + b^2 \leq 1 \quad \text{and then } \lim_{n\to\infty} A^{2n} = \begin{cases} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \text{if } a^2 + b^2 = 1 \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{if } a^2 + b^2 < 1 \end{cases}}.$$

Problem 10. (10 points) Find all 3×3 matrices A with all real entries such that

$$\mathsf{Nul}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 + 2x_2 - x_3 = 0 \right\}.$$

Solution:

We are looking for matrices A such that

$$\mathsf{Nul}(A) = (\mathsf{Col}(A^\top))^\perp = \left(\mathbb{R}\text{-span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\} \right)^\perp.$$

Taking the $\perp$ of both sides gives

$$\mathsf{Col}(A^\top) = \mathbb{R}\text{-span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}.$$

This means that $A^\top = \begin{bmatrix} a & b & c \\ 2a & 2b & 2c \\ -a & -b & -c \end{bmatrix}$ for some $a, b, c \in \mathbb{R}$ that are not all zero.

So our answer is

$$\boxed{A = \begin{bmatrix} a & 2a & -a \\ b & 2b & -b \\ c & 2b & -c \end{bmatrix} \text{ for } a, b, c \in \mathbb{R} \text{ with } |a| + |b| + |c| > 0}.$$

Problem 11. (10 points = 5 + 5)

This problem has 2 parts.

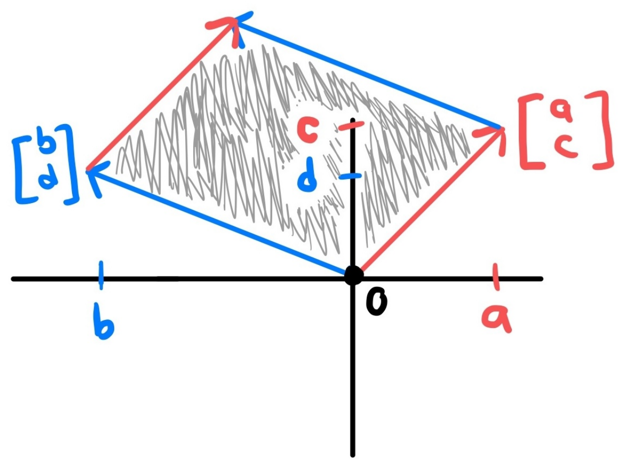
(a) Suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$. Draw a picture that represents the region

$$\left\{ Ax \in \mathbb{R}^2 : x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \text{ has } 0 \leq x_1 \leq 1 \text{ and } 0 \leq x_2 \leq 1 \right\}.$$

Your picture should indicate the correct shape of this region, and you should label the features of your picture that depend on a, b, c, and d.

Solution:

The picture should look something like this:



The important details are to show a parallelogram with one vertex at the origin, and with the sides incident to 0 labeled by $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$.

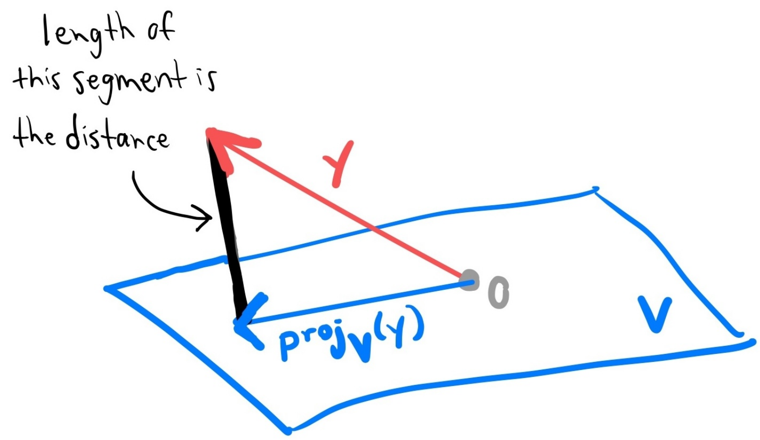
The orientation of the picture does not need to match the example shown.

(b) Suppose V is a 2-dimensional subspace of $\mathbb{R}^3$ and $y \in \mathbb{R}^3$ has $y \notin V$.

Draw a generic picture that shows
- the subspace V,
- the vector y,
- the orthogonal projection $\text{proj}_V(y)$, and
- the line segment whose length is the distance from y to V.

Solution:

The picture should look something like this:



Problem 12. (10 points) Find all real numbers x such that the vectors

$$\begin{bmatrix} x \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ x \\ 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ x \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ x \end{bmatrix}$$

do **not** form a basis for $\mathbb{R}^4$. For each of the values of x that you find, compute the dimension of the subspace of $\mathbb{R}^4$ that the vectors span.

Solution:

If $x = 1$ then the four vectors are equal and span a 1-dimensional subspace.

If $x \neq 1$ then the four vectors must span a subspace of dimension 3 when they do not form a basis, since the first three vectors are linearly independent as

$$\mathsf{RREF}\begin{bmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \\ 1 & 1 & 1 \end{bmatrix} = \mathsf{RREF}\begin{bmatrix} x-1 & 0 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \\ 1 & 1 & 1 \end{bmatrix} = \mathsf{RREF}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and the last matrix has a pivot in every column.

This happens if $x = -3$ as then the vectors sum to $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ so are linearly dependent.

Assume $x \neq 1$ and $x \neq -3$. Then we can row reduce

$$\begin{bmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{bmatrix} \to \begin{bmatrix} x+3 & x+3 & x+3 & x+3 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & x-1 & 0 & 0 \\ 0 & 0 & x-1 & 0 \\ 0 & 0 & 0 & x-1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and this tells us that the four vectors are a basis for $\mathbb{R}^4$.

So our final answer is: the vectors are not a basis when either

- $\boxed{x = 1}$ and then the spanned subspace has dimension $\boxed{1}$; or
- $\boxed{x = -3}$ and then the spanned subspace has dimension $\boxed{3}$.

Problem 13. (10 points)

Suppose $u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in \mathbb{R}^3$ and $w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \in \mathbb{R}^3$ satisfy

$$\det \begin{bmatrix} u_1 & w_1 & 1 \\ u_2 & w_2 & 0 \\ u_3 & w_3 & 0 \end{bmatrix} = 1, \quad \det \begin{bmatrix} u_1 & w_1 & 0 \\ u_2 & w_2 & 1 \\ u_3 & w_3 & 0 \end{bmatrix} = 2, \quad \text{and} \quad \det \begin{bmatrix} u_1 & w_1 & 0 \\ u_2 & w_2 & 0 \\ u_3 & w_3 & 1 \end{bmatrix} = 3.$$

Find a basis for the subspace $\mathbb{R}\text{-span}\{u, w\}$.

Solution:

The given determinants tell us that u and w are linearly independent so $\mathbb{R}\text{-span}\{u, w\}$ is 2-dimensional. Since

$$\det \begin{bmatrix} u_1 & w_1 & 2 \\ u_2 & w_2 & -1 \\ u_3 & w_3 & 0 \end{bmatrix} = 2 \det \begin{bmatrix} u_1 & w_1 & 1 \\ u_2 & w_2 & 0 \\ u_3 & w_3 & 0 \end{bmatrix} - \det \begin{bmatrix} u_1 & w_1 & 0 \\ u_2 & w_2 & 1 \\ u_3 & w_3 & 0 \end{bmatrix} = 2 \cdot 1 - 2 = 0$$

we deduce that $\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \in \mathbb{R}\text{-span}\{u, w\}$, and since

$$\det \begin{bmatrix} u_1 & w_1 & 0 \\ u_2 & w_2 & 3 \\ u_3 & w_3 & -2 \end{bmatrix} = 3 \det \begin{bmatrix} u_1 & w_1 & 0 \\ u_2 & w_2 & 1 \\ u_3 & w_3 & 0 \end{bmatrix} - 2 \det \begin{bmatrix} u_1 & w_1 & 0 \\ u_2 & w_2 & 0 \\ u_3 & w_3 & 1 \end{bmatrix} = 3 \cdot 2 - 2 \cdot 3 = 0$$

we deduce that $\begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \in \mathbb{R}\text{-span}\{u, w\}$. The vectors

$$\boxed{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}}$$

are linearly independent so they must form a basis for $\mathbb{R}\text{-span}\{u, w\}$.

Problem 14. (10 points) Suppose A is a 3×3 matrix with

$$\mathsf{trace}(A) = p, \quad \mathsf{trace}(A^2) = q, \quad \text{and} \quad \mathsf{trace}(A^3) = r.$$

Derive an expression for $\det(A)$ in terms of p, q, and r.

Justify your answer to receive full credit.

Solution:

We are looking for a formula that holds for all 3×3 matrices, so to try to find the answer we should first assume A has some simpler form like that of a triangular matrix, say with diagonal entries x, y, and z. Then

$$p = x + y + z, \quad q = x^2 + y^2 + z^2, \quad \text{and} \quad r = x^3 + y^3 + z^3$$

and $\det(A) = xyz$. The only way to get an xyz term is by cubing p:

$$p^3 = (x^3 + y^3 + z^3) + 3(xy^2 + x^2y + xz^2 + xz^2 + yz^2 + y^2z) + 6xyz.$$

But now we need to cancel the other terms. The only other way to get the second group of terms is from the product

$$pq = (x^3 + y^3 + z^3) + (xy^2 + x^2y + xz^2 + xz^2 + yz^2 + y^2z).$$

Now we can just calculate

$$p^3 - 3pq = -2(x^3 + y^3 + z^3) + 6xyz = 6xyz - 2r.$$

Thus $6xyz = p^3 - 3pq + 2r$ and $xyz = \frac{1}{6}\left(p^3 - 3pq + 2r\right)$ so

$$\boxed{\det(A) = \tfrac{1}{6}p^3 - \tfrac{1}{2}pq + \tfrac{1}{3}r}.$$

Why does this formula hold for 3×3 matrices rather than just triangular ones? This can either be checked directly by a tedious computation on setting

$$A = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{bmatrix},$$

or we can use the fact that every such matrix A is similar to a triangular one, and the values of p, q, and r, don't change if we replace A by PAP^{-1}.