FINAL EXAMINATION – MATH 2121, FALL 2023

Name:

Student ID:

Email:

Tutorial: T1A T1B T1C T2A T2B T2C

Problem #	Points Possible	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
11	10	
12	10	
13	10	
14	10	
15	10	
16	10	
17	10	
Total	180	

You have **180 minutes** to complete this exam.

No books, notes, or electronic devices can be used during the test.

RECOMMENDED: It will help us to grade your solutions if you draw a $\boxed{\text{box}}$ around your answers to computational questions. If we cannot determine what your answer is, you may lose points. Partial credit can be given on some problems.

Good luck!

Problem 1. (20 points) Warmup: definitions and core concepts.

Provide short answers to the following questions.

(1) What is the definition of a **linear** function $f : \mathbb{R}^n \to \mathbb{R}^m$?

(2) How many solutions can a linear system have?

(3) What is the definition of a **subspace** of a vector space?

(4) How can you compute the rank of an $m \times n$ matrix A?

(5) How can you compute the inverse of an invertible $n \times n$ matrix A?

(6) What region of $\mathbb{R}^2$ always has area equal to $\pm \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$? Draw and label a picture that represents this region.

(7) The least-squares solutions to $Ax = b$ are the exact solutions to what matrix equation?

(8) What $n \times n$ matrices have n orthonormal eigenvectors?

(9) What is the definition of the **singular values** of a matrix A?

(10) Suppose A is a $2 \times n$ matrix with a singular value decomposition

$$A = U\Sigma V^\top.$$

Assume $\operatorname{rank} A = 2$. Describe the shape

$$\{Av \in \mathbb{R}^2 : v \in \mathbb{R}^n \text{ with } \|v\| = 1\}$$

and explain how this shape is related to the matrices U and Σ.

Problem 2. (10 points) Suppose a and b are real numbers. Consider the lines

$$L_1 = \left\{ \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in \mathbb{R}^2 : v_2 = av_1 \right\} \quad \text{and} \quad L_2 = \left\{ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^2 : w_2 = bw_1 \right\}.$$

For which values of a and b is there exactly one way of writing

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} = v + w$$

where $v \in L_1$ and $w \in L_2$? Find a formula for v and w in this case.

Solution:

Problem 3. (10 points) Does there exist a pair of 2×2 matrices A and B with all real entries such that A has only one real eigenvalue, B has only one real eigenvalue, and $A + B$ has two distinct real eigenvalues?

Find an example or explain why none exists.

Solution:

Problem 4. (10 points) Let V be the vector space of polynomials $f = ax^2 + bx + c$ of degree at most two with all coefficients $a, b, c \in \mathbb{R}$. Given $f, g \in V$ let

$$f \bullet g = \int_0^1 fg.$$

Here we define integration $\int_0^1$ to be the linear operation on polynomials with

$$\int_0^1 x^n = 1/(n+1).$$

Find a basis for V that is orthonormal using this definition of inner product.

In other words, if $d = \dim V$, then find a basis $f_1, f_2, \ldots, f_d$ for V such that

$$f_i \bullet f_i = 1 \quad \text{and} \quad f_i \bullet f_j = 0 \quad \text{for all } i, j \in \{1, 2, \ldots, d\} \text{ with } i \neq j.$$

Solution:

Problem 5. (10 points) Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a one-to-one linear transformation with standard matrix A. If all we know is that $n \in \{1, 2, 3\}$ and $m \in \{1, 2, 3\}$, then what matrices could occur as $\mathsf{RREF}(A)$?

Describe the possibilities for $\mathsf{RREF}(A)$ in as much detail as you can.

Solution:

Problem 6. (10 points) Suppose $A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 0 \\ 2 & 3 & 4 \end{bmatrix}$ and $v \in \mathbb{R}^3$.

Define w_i to be the determinant of A with column i replaced by v.

Does any matrix B exist with $Bv = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ for all choices of $v \in \mathbb{R}^3$?

Compute the matrix B or explain why no such B exists.

Solution:

Problem 7. (10 points) Suppose $u, v, w \in \mathbb{R}^3$ and $A = \begin{bmatrix} u & v & w \end{bmatrix}$.

If $\det(A) = 30$ then what is

$$\det \begin{bmatrix} u + 2v - 3w & v + w & 2u + v + 2w \end{bmatrix}?$$

Justify your answer to receive full credit.

Solution:

Problem 8. (10 points) Is the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

diagonalizable over the complex numbers?

If it is not, then explain why not. If it is, then find an invertible matrix P and a diagonal matrix D, possibly with complex entries, such that $A = PDP^{-1}$.

Solution:

Problem 9. (10 points) Let A be a symmetric $n \times n$ matrix with exactly one nonzero position in each row and exactly one nonzero position in each column.

Suppose the nonzero positions of A that are on or above the diagonal are

$$(i_1, j_1), (i_2, j_2), \dots, (i_k, j_k).$$

In terms of this data, describe an orthogonal basis for $\mathbb{R}^n$ that consists of eigenvectors for A. Be as concrete as possible.

Solution:

Problem 10. (10 points) Suppose A is a 3×3 matrix with all real entries, whose eigenvalues include the complex numbers 2 and $1 - i$. Find a polynomial formula for the function $f(x) = \det(A^{-1} - xI)$ and compute $f(5)$.

Solution:

Problem 11. (10 points) Suppose $u, v, w \in \mathbb{R}^n$ are linearly independent vectors.

For which values of $c \in \mathbb{R}$ are the three vectors

$$5w - 3u, \quad 5u + 3v + 4w, \quad 6v - 2u + cw$$

linearly dependent?

Solution:

Problem 12. (10 points) Suppose $u, v, w \in \mathbb{R}^5$.

What are the possible values of $\mathrm{rank}(uu^\top + vv^\top + ww^\top)$?

Justify your answer to receive full credit.

Solution:

Problem 13. (10 points) A is a 2×2 matrix and $-1 < \lambda < 1$ is a real number with

$$A \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}.$$

Compute A and $\lim_{n\to\infty} A^n$.

The entries in your answer for A should be expressions involving λ.

Solution:

Problem 14. (10 points) Suppose

$$v = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find a matrix A such that if $x \in \mathbb{R}^4$ then

$$Ax = (\text{ the vector in } \mathbb{R}\text{-span}\{v, w\} \text{ that is as close to } x \text{ as possible }).$$

Solution:

Problem 15. (10 points) A is an invertible $n \times n$ matrix with at least one real eigenvalue. There is no nonzero vector $v \in \mathbb{R}^n$ such that $Av = v$. If 3 is the only eigenvalue of $A + 2A^{-1}$ then what number must be an eigenvalue of A? Justify your answer to receive full credit.

Solution:

Problem 16. (10 points) Does there exist an invertible 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

for which the determinant of every 2×2 submatrix involving **consecutive** rows and columns is zero? In other words, with

$$\det \begin{bmatrix} a_{ij} & a_{i,j+1} \\ a_{i+1,j} & a_{i+1,j+1} \end{bmatrix} = 0$$

for all $i \in \{1, 2\}$ and $j \in \{1, 2\}$?

Find an example or explain why none exists.

Solution:

Problem 17. (10 points) What is the largest possible number that can occur as the determinant of a 3×3 matrix with all entries in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$? What matrix achieves this determinant?

Solution: