MIDTERM SOLUTIONS – MATH 2121, FALL 2024

Problem 1. (10 points)

Find the general solution to the linear system

$$\begin{cases} x_1 + 3x_2 - 5x_3 = 4 \\ x_1 + 4x_2 - 8x_3 = 7 \\ -3x_1 - 7x_2 + 9x_3 = -6. \end{cases}$$

Solution:

The augmented matrix is $A = \left[\begin{array}{rrr|r} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{array}\right]$ which we row reduce as

$$\begin{aligned} A = \left[\begin{array}{rrr|r} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{array}\right] &\to \left[\begin{array}{rrr|r} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ -3 & -7 & 9 & -6 \end{array}\right] \\ &\to \left[\begin{array}{rrr|r} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{array}\right] \\ &\to \left[\begin{array}{rrr|r} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \\ &\to \left[\begin{array}{rrr|r} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] = \mathsf{RREF}(A). \end{aligned}$$

The last column does not have a pivot, so the system has at least one solution.
The pivot columns are columns 1 and 2.
Therefore the basic variables are x_1 and x_2, while x_3 is a free varaible.
The linear system with augmented matrix $\mathsf{RREF}(A)$ is

$$\begin{cases} x_1 + 4x_3 = -5 \\ x_2 - 3x_3 = 3 \\ 0 = 0 \end{cases} \qquad \text{which we rewrite as} \qquad \begin{cases} x_1 = -5 - 4x_3 \\ x_2 = 3 + 3x_3. \end{cases}$$

Thus the general solution is

$$\boxed{(x_1, x_2, x_3) = (-5 - 4a, 3 + 3a, a) \text{ for all real numbers } a \in \mathbb{R}}.$$

Problem 2. (10 points)

Compute how many 2×4 matrices A exist with all of the following properties:

- each entry of A is equal to 0 and 1,
- A is in reduced echelon form, and
- A is the augmented matrix of a consistent linear system.

Justify your answer.

Solution:

The last condition means A cannot have a pivot in the last column.

If A has no pivot positions then A must be the zero matrix:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This is the augmented matrix of a consistent linear system.
So it contributes 1 matrix to the answer.

If A has exactly one pivot position then A must have the form

$$\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where each $*$ can be 0 or 1.
This contributes $8 + 4 + 2 = 14$ matrices to the answer.

Finally if A has exactly two pivot positions then A must have the form

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{bmatrix} \text{ or } \begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{bmatrix}$$

where each $*$ can be 0 or 1.
This contributes $16 + 8 + 4 = 28$ matrices to the answer.

Adding things up gives $1 + 14 + 28 = \boxed{43}$ matrices.

Problem 3. (10 points)

Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation satisfying

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

We are told that at least 2 of the columns of the standard matrix of T are equal to each other as vectors. Find all possibilities for the standard matrix of T.

Solution:

Let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Suppose $T(e_3) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Then $T(e_2 + 2e_3) = T(e_2) + 2T(e_3) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ so

$$T(e_2) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2-2x \\ -2y \\ 1-2z \end{bmatrix}.$$

Next, we have $T(e_1 + 2e_2) = T(e_1) + 2T(e_2) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ so

$$T(e_1) = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 2-2x \\ -2y \\ 1-2z \end{bmatrix} = \begin{bmatrix} 4x-4 \\ 1+4y \\ 4z \end{bmatrix}.$$

Thus the standard matrix of T in terms of x, y, and z is

$$A = \begin{bmatrix} 4x-4 & 2-2x & x \\ 1+4y & -2y & y \\ 4z & 1-2z & z \end{bmatrix}.$$

We are told that two columns are equal. If columns 1 and 2 are equal then

$$\begin{cases} 4x-4 = 2-2x \\ 1+4y = -2y \\ 4z = 1-2z \end{cases}$$

so we must have

$$x = 1, \quad y = -1/6, \quad \text{and} \quad z = 1/6 \quad \Rightarrow \quad \boxed{A = \begin{bmatrix} 0 & 0 & 1 \\ 1/3 & 1/3 & -1/6 \\ 2/3 & 2/3 & 1/6 \end{bmatrix}}.$$

If columns 1 and 3 are equal then similarly

$$x = 4/3, \quad y = -1/3, \quad z = 0 \quad \Rightarrow \quad \boxed{A = \begin{bmatrix} 4/3 & -2/3 & 4/3 \\ -1/3 & 2/3 & -1/3 \\ 0 & 1 & 0 \end{bmatrix}}.$$

Finally if columns 2 and 3 are equal then

$$x = 2/3, \quad y = 0, \quad z = 1/3 \quad \Rightarrow \boxed{A = \begin{bmatrix} -4/3 & 2/3 & 2/3 \\ 1 & 0 & 0 \\ 4/3 & 1/3 & 1/3 \end{bmatrix}}.$$

Problem 4. (10 points)

Let $a_1, a_2, \dots, a_n$ be any real numbers. Define A to be the lower-triangular $n \times n$ matrix whose entry in position (i,j) is a_j whenever $i \geq j$. This means A looks like

$$A = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_1 & a_2 & 0 & & 0 \\ a_1 & a_2 & a_3 & & 0 \\ \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}.$$

Determine when A is invertible and describe A^{-1}.

Be as explicit as possible and justify your answer.

Solution:

The matrix is invertible when its determinant is nonzero.
The matrix is triangular so $\det(A) = a_1 a_2 \cdots a_n$.
Hence the matrix is invertible when $\boxed{a_1, a_2, \dots, a_n \text{ are all nonzero.}}$
To find A^{-1} we want to row reduce the $n \times 2n$ matrix

$$\left[\begin{array}{c|c} A & I \end{array}\right] = \left[\begin{array}{ccccc|ccccc} a_1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ a_1 & a_2 & 0 & & 0 & 0 & 1 & 0 & & 0 \\ a_1 & a_2 & a_3 & & 0 & 0 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_n & 0 & 0 & 0 & \cdots & 1 \end{array}\right].$$

Consider the following row operators. First subtract one copy of the first row from all lower rows. This will create zeros below position $(1,1)$ and -1's below position $(1, n+1)$:

$$\left[\begin{array}{ccccc|ccccc} a_1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & & 0 & -1 & 1 & 0 & & 0 \\ 0 & a_2 & a_3 & & 0 & -1 & 0 & 1 & & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ 0 & a_2 & a_3 & \cdots & a_n & -1 & 0 & 0 & \cdots & 1 \end{array}\right].$$

Next, subtract one copy of the second row from all lower rows. This will create zeros below positions $(2,2)$ and $(2, n+1)$, while keeping -1 in position $(2, n+1)$, and also creating -1's below position $(2, n+2)$:

$$\left[\begin{array}{ccccc|ccccc} a_1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & & 0 & -1 & 1 & 0 & & 0 \\ 0 & 0 & a_3 & & 0 & 0 & -1 & 1 & & 0 \\ \vdots & & & \ddots & \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & a_3 & \cdots & a_n & 0 & -1 & 0 & \cdots & 1 \end{array}\right].$$

If we continue in this way, by subtracting one copy of row i from all lower rows for $i = 3, 4, 5, \ldots, n-1$, then we can row reduce $\left[\begin{array}{c|c} A & I \end{array}\right]$ to

$$\left[\begin{array}{ccccc|ccccc} a_1 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & & 0 & -1 & 1 & 0 & & 0 \\ 0 & 0 & a_3 & & 0 & 0 & -1 & 1 & & 0 \\ \vdots & & & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n & 0 & \cdots & 0 & -1 & 1 \end{array}\right].$$

In this picture, the first n columns are the diagonal matrix $\mathsf{diag}(a_1, a_2, a_3, \ldots, a_n)$ with a_i in position (i, i), while the last n columns are the identity matrix but with -1 in each position just below the diagonal. To convert this to reduced echelon form, we need to divide row i by a_i for each $i = 1, 2, \ldots, n$. Then the last n columns will give

$$\boxed{A^{-1} = \begin{bmatrix} a_1^{-1} & & & & & \\ -a_2^{-1} & a_2^{-1} & & & & \\ & -a_3^{-1} & a_3^{-1} & & & \\ & & -a_4^{-1} & a_4^{-1} & & \\ & & & \ddots & \ddots & \\ & & & & -a_n^{-1} & a_n^{-1} \end{bmatrix}}$$

where all blank entries are zero. This means the matrix that has a_i^{-1} in position (i, i) for all $i = 1, 2, \ldots, n$, along with $-a_i^{-1}$ in position $(i, i-1)$ for all $i = 2, 3, \ldots, n$, and with zero in all other positions.

Problem 5. (10 points)

Consider the three matrices

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -4 & 5 & 8 & -4 \\ 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \end{bmatrix}, \quad C = \begin{bmatrix} -7 & 1 & -1 & 6 & -3 \\ -1 & 3 & 2 & -2 & 1 \\ -4 & 8 & 5 & -4 & 2 \end{bmatrix}.$$

The first matrix A has reduced echelon form

$$\mathsf{RREF}(A) = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This question has three parts:

(a) Find a basis for $\mathrm{Col}(A)$, a basis for $\mathrm{Nul}(A)$, and the rank of A.

Solution:

A basis for $\mathrm{Col}(A)$ consists of the pivot columns of A given by $\boxed{\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}}$.

The linear system corresponding to $\mathsf{RREF}(A)x = 0$ can be written as

$$\begin{cases} x_1 = 2x_2 + x_4 - 3x_5 \\ x_3 = -2x_4 + 2x_5 \end{cases}$$

so if $x \in \mathrm{Nul}(A)$ then

$$x = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

which means a basis for $\mathrm{Nul}(A)$ is given by $\boxed{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$.

Finally the rank of A is $\boxed{2}$.

(b) Find a basis for $\mathrm{Col}(B)$, a basis for $\mathrm{Nul}(B)$, and the rank of B.

Solution:

B is row equivalent to A so its reduced echelon form is the same.

So a basis for $\mathrm{Col}(B)$ consists of the pivot columns of B given by $\boxed{\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}}$.

Row equivalent matrices have the same null space.

Therefore a basis for $\mathrm{Nul}(B)$ is again $\boxed{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$.

The rank of B is again $\boxed{2}$.

(c) Find a basis for $\mathrm{Col}(C)$, a basis for $\mathrm{Nul}(C)$, and the rank of C.

Solution:

The matrix C is obtained by reversing the order of the columns of A.

This means $\mathrm{Col}(C) = \mathrm{Col}(A)$, so one basis for $\mathrm{Col}(C)$ is $\boxed{\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}}$.

This also means $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \mathrm{Nul}(C)$ if and only if $\begin{bmatrix} x_5 \\ x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix} \in \mathrm{Nul}(A)$.

So a basis for $\mathrm{Nul}(C)$ is obtained by flipping the basis of $\mathrm{Nul}(A)$:

$$\boxed{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ -3 \end{bmatrix}}.$$

The rank of C is also $\boxed{2}$.

Problem 6. (10 points)

Many algorithms in this course extract information about a matrix A from properties of its reduced echelon form $\mathsf{RREF}(A)$.

Explain how you can use $\mathsf{RREF}(A)$ to answer the following 5 questions.

Write your answers in the form "YES if (...some condition ...), NO otherwise".

(a) Are the columns of A linearly independent?

Solution:

YES if there is a pivot position in every column of $\mathsf{RREF}(A)$.

(b) Is A the coefficient matrix of any inconsistent linear system?

Solution:

YES if there is a row of zeros in $\mathsf{RREF}(A)$,
or equivalently if there are not pivots in every row of $\mathsf{RREF}(A)$.

(c) Is A invertible?

Solution:

YES if $\mathsf{RREF}(A)$ is the identity matrix.

(d) Is A row equivalent to another matrix B?

Solution:

YES if $\mathsf{RREF}(A) = \mathsf{RREF}(B)$.

(e) Does A have full rank?

Solution:

YES if the number of pivots in $\mathsf{RREF}(A)$ is equal to whichever is smaller between the number of rows or columns of A.

Problem 7. (10 points) Let $A = \begin{bmatrix} 2 & 4 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 7 & 2 \\ 2 & 1 \end{bmatrix}$.

Find a basis for the intersection of $\mathrm{Col}(A)$ and $\mathrm{Col}(B)$.

Solution:

The vectors in $\mathrm{Col}(A)\cap\mathrm{Col}(B)$ are the vectors that can be expressed as $Ax = By$ for some $x, y \in \mathbb{R}^2$, in which case $Ax - By = 0$. This means $Ax \in \mathrm{Col}(A) \cap \mathrm{Col}(B)$ precisely when there exists a solution to the matrix equation

$$\begin{bmatrix} A & -B \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = 0.$$

We can solve this equation by finding a basis for the null space of $M = \begin{bmatrix} A & -B \end{bmatrix}$. We row reduce this matrix as

$$M = \begin{bmatrix} 2 & 4 & -1 & -1 \\ 0 & 1 & -7 & -2 \\ 2 & 0 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 4 & 1 & 0 \\ 0 & 1 & -7 & -2 \\ 2 & 0 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 29 & 8 \\ 0 & 1 & -7 & -2 \\ 2 & 0 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 & 8/29 \\ 0 & 1 & -7 & -2 \\ 2 & 0 & -2 & -1 \end{bmatrix}$$
$$\to \begin{bmatrix} 0 & 0 & 1 & 8/29 \\ 0 & 1 & 0 & 56/29-2 \\ 2 & 0 & 0 & 16/29-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 8/29 \\ 0 & 1 & 0 & -2/29 \\ 2 & 0 & 0 & -13/29 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & -13/58 \\ 0 & 1 & 0 & -2/29 \\ 0 & 0 & 1 & 8/29 \end{bmatrix} = \mathsf{RREF}(M).$$

This tells us that if $\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} \in \mathrm{Nul}(M)$ then $\begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} = y_2 \begin{bmatrix} 13/58 \\ 2/29 \\ -8/29 \\ 1 \end{bmatrix}$

Thus $\begin{bmatrix} 13/58 \\ 2/29 \\ -8/29 \\ 1 \end{bmatrix}$ is a basis for this null space.

In fact, a nicer basis for $\mathrm{Nul}(M)$ is given by the rescaled vector $\begin{bmatrix} 13 \\ 4 \\ -16 \\ 58 \end{bmatrix}$.

This means that any vector in $\mathrm{Col}(A) \cap \mathrm{Col}(B)$ has the form

$$Ax = By \quad \text{where } x = c\begin{bmatrix} 13 \\ 4 \end{bmatrix} \text{ and } y = c\begin{bmatrix} -16 \\ 58 \end{bmatrix} \text{ for some scalar } c \in \mathbb{R}.$$

In other words, $\mathrm{Col}(A) \cap \mathrm{Col}(B)$ is also one-dimensional, so any of its nonzero elements is a basis, and one such nonzero element is

$$A\begin{bmatrix} 13 \\ 4 \end{bmatrix} = B\begin{bmatrix} -16 \\ 58 \end{bmatrix} = \begin{bmatrix} 26+16 \\ 24 \\ 26 \end{bmatrix} = \boxed{\begin{bmatrix} 42 \\ 4 \\ 26 \end{bmatrix}}.$$

Problem 8. (10 points) Alice and Bob play the following game: they start with an empty 4×4 matrix A and take turns writing numbers in each of the 16 positions.

Once the matrix A is filled, Bob wins if $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \in \mathrm{Nul}(A)$ and Alice wins otherwise.

If Alice goes first, then which player (if any) has a winning strategy?

What is this strategy?

Solution:

$\boxed{\text{Bob can always win}}$. Here is one possible strategy.

Each time Alice puts a number a in position (i, j), Bob chooses another free position (i, k) in the same row (which is possible since there are an even number of columns, and Alice goes first) and puts the number $-\frac{j}{k}a$ in that position.

Then in the final matrix A, the two positions (i, j) and (i, k) will contribute

$$a \cdot j + (-\tfrac{j}{k}a) \cdot k = aj - aj = 0$$

to the value in row i of $A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$. The value in row i of $A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ will be the sum of two such contributions, both zero, and so will be zero. This means that under the given strategy, no matter what Alice does, we will have

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \in \mathrm{Nul}(A).$$

Problem 9. (10 points) What is the greatest common divisor of the numbers that occur as $\det(A)$ as A ranges over all invertible 4×4 matrices whose entries are each equal to $+1$ or -1?

The *greatest common divisor* of a set of nonzero integers is defined to be the largest positive integer that divides every integer in the set.

Justify your answer.

Solution:

Replacement row operations do not change the determinant so

$$\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} = 8.$$

Thus the GCD must be at most 8.

We claim that the GCD is actually 8, because 8 divides the determinant of every matrix we are considering. One way to see this is as follows.

Let A be a 4×4 invertible matrix with all entries equal to -1 or $+1$. Form B by adding the first row of A to rows 2, 3, and 4. Then $\det A = \det B$.

Every entry of B in row 1 is $+1$ or -1, just as in A. However, below row 1, each entry of B is either 0, $+2$, or -2. Therefore in the formula

$$\det B = \sum_{X \in S_n} \mathsf{prod}(X, B)(-1)^{\mathsf{inv}(X)}$$

each coefficient $\mathsf{prod}(X, B)$ is a product of four numbers: ± 1 from the first row, then three numbers which are each 0 or ± 2 from the other rows.

Such a product is either zero or ± 8, and so each term $\mathsf{prod}(X, B)(-1)^{\mathsf{inv}(X)}$ for $X \in S_n$ is equal to zero or ± 8.

Thus $\det B$ can be written as a sum of numbers equal to ± 8, which is automatically divisible by 8. As $\det A = \det B$, the determinant of A is divisible by 8.

We conclude that the desired GCD is $\boxed{8}$.