This document is an **exact transcript** of the lecture, with extra summary and vocabulary sections for your convenience. Due to time constraints, the lectures sometimes only contain limited illustrations, proofs, and examples. For a more thorough discussion of the course content, **consult the textbook**.

Summary

Quick summary of today's notes. Lecture starts on next page.

- Let n be a positive integer and let A and B be $n \times n$ matrices.
- It always holds that $\det A = \det A^{\top}$.
- If A is invertible then det $A \neq 0$. If A is not invertible then det A = 0.
- It always holds that $\det AB = (\det A)(\det B)$.
- A matrix is *triangular* if it looks like

ſ	*	*	*	* -		*	0	0	0]
	0	*	*	*	0.1	*	*	0	0
	0	0	*	*	OI	*	*	*	0
	0	0	0	*		*	*	*	*

where the *'s are arbitrary entries.

Let $a_{ij} \in \mathbb{R}$ denote the entry of A in the *i*th row and *j*th column.

If A is triangular then $\det A = a_{11}a_{22}a_{33}\cdots a_{nn}$ = the product of the diagonal entries of A. The matrix A is *diagonal* if $a_{ij} = 0$ whenever $i \neq j$. Diagonal matrices are triangular.

- Here is an algorithm to compute det A:
 - Perform a series of row operations to transform A to a matrix E in echelon form.
 - Keep track of a scalar denom $\in \mathbb{R}$ as you do this. Start with denom = 1.
 - Whenever you swap two rows of A, multiply denom by -1.
 - Whenever you multiply a row of A by a nonzero number, multiply denom by that number.

– Thon	dot $A = \frac{\det E}{\det E}$	product of diagonal entries of E
- 1 11611	det A = denom	denom

• Here is another way to compute $\det A$.

Again write a_{ij} for the entry of A in row i and column j.

Also let $A^{(i,j)}$ be the matrix formed from A by deleting row i and column j.

Then
$$\det A = a_{11} \det A^{(1,1)} - a_{12} \det A^{(1,2)} + a_{13} \det A^{(1,3)} - \dots - (-1)^n a_{1n} \det A^{(1,n)}$$

This formula is complicated and inefficient for generic matrices.

It is useful when many entries of A are equal to zero, since then the formula has few terms.

Also, when $n \leq 3$ and you expand all the terms in this formula, you get the identities

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \quad \text{and} \quad \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg).$$

1 Last time: introduction to determinants

Let n be a positive integer.

A *permutation matrix* is a square matrix formed by rearranging the columns of the identity matrix.

Equivalently, a permutation matrix is a square matrix whose entries are all 0 or 1, and that has exactly one nonzero entry in each row and in each column.

Let S_n be the set of $n \times n$ permutation matrices.

If A is an $n \times n$ matrix and $X \in S_n$, then AX has the same columns as A but in a different order.

The columns of A are "permuted" by X to form AX.

Example. The six elements of S_3 are

$\left[\begin{array}{rrrr} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$	$\left[\begin{array}{rrrr} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right] \left[\begin{array}{r} 0 \\ 1 \\ 0 \end{array}\right]$	$ \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} $	$\left[\begin{array}{rrrr} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right] \left[\begin{array}{rrrr} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array}\right]$
Notice that $\begin{bmatrix} a & b \\ d & e \\ g & h \end{bmatrix}$	$\begin{bmatrix} c \\ f \\ i \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} =$	$= \left[\begin{array}{ccc} b & c & a \\ e & f & d \\ h & i & g \end{array} \right].$	

Given $X \in S_n$ and an arbitrary $n \times n$ matrix A:

- Define $\operatorname{prod}(X, A)$ to be the product of the entries of A in the nonzero positions of X.
- Define inv(X) to be the number of 2×2 submatrices of X equal to $\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$.

To form a 2×2 submatrix of X, choose any two rows and any two columns, **not necessarily adjacent**, and then take the 4 entries determined by those rows and columns.

Each 2×2 submatrix of a permutation matrix is either

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Example. prod $\left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \right) = cdh$

Example. inv
$$\left(\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right) = 2$$
 and inv $\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$ and inv $\left(\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right) = 3.$

Definition. The *determinant* of an $n \times n$ matrix A is $\det A = \sum_{X \in S_n} \operatorname{prod}(X, A)(-1)^{\operatorname{inv}(X)}$

This general formula simplifies to the following expressions for n = 1, 2, 3:

$$\det \begin{bmatrix} a \end{bmatrix} = a, \quad \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc, \quad \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg).$$

For $n \ge 4$, our formula for det A is a sum with at least 24 terms, which is not easy to compute by hand (or with a computer, for slightly larger n). We will describe a better way to compute determinants today.

The most important properties of the determinant are described by the following theorem:

Theorem. The determinant is the unique function det : $\{n \times n \text{ matrices}\} \rightarrow \mathbb{R}$ with these 3 properties:

- (1) $\det I_n = 1$.
- (2) If B is formed by switching two columns in an $n \times n$ matrix A, then $\det A = -\det B$.
- (3) Suppose A, B, and C are $n \times n$ matrices with columns

 $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix} \text{ and } C = \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix}.$ Assume that $a_i = pb_i + qc_i$ for numbers $p, q \in \mathbb{R}$. Assume also that $a_j = b_j = c_j$ for all other indices $i \neq j \in \{1, 2, \dots, n\}.$ Then $\boxed{\det A = p \det B + q \det C}.$

Remark. Our formulation of this theorem last time required i = 1 in property (3). However, we showed that this property combined with (2) implies the more general version of (3) described here.

Corollary. If A is a square matrix that is not invertible then det A = 0.

Corollary. If A is a permutation matrix then det $A = (-1)^{inv(A)}$.

Proof. $\operatorname{prod}(X,Y) = 0$ if X and Y are different $n \times n$ permutation matrices, but $\operatorname{prod}(X,X) = 1$. \Box

2 More properties of the determinant

Recall that A^{\top} denotes the transpose of a matrix A (the matrix whose rows are the columns of A).

Lemma. If $X \in S_n$ then $X^{\top} \in S_n$ and $inv(X) = inv(X^{\top})$.

Proof. Transposing a permutation matrix does not affect the # of 2×2 submatrices equal to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. \Box

Corollary. If A is any square matrix then det $A = det(A^{\top})$.

Proof. If $X \in S_n$ then $\operatorname{prod}(X, A) = \operatorname{prod}(X^{\top}, A^{\top})$, so our formula for the determinant gives

$$\det A = \sum_{X \in S_n} \operatorname{prod}(X, A)(-1)^{\operatorname{inv}(X)} = \sum_{X \in S_n} \operatorname{prod}(X^\top, A^\top)(-1)^{\operatorname{inv}(X^\top)}$$

As X ranges over all elements of S_n , the transpose X^{\top} also ranges over all elements of S_n . The second sum is therefore equal to $\sum_{X \in S_n} \operatorname{prod}(X, A^{\top})(-1)^{\operatorname{inv}(X)} = \det(A^{\top})$.

Corollary. If A is a square matrix with two equal rows then $\det A = 0$.

Proof. In this case A^{\top} has two equal columns, so $0 = \det A^{\top} = \det A$.

The following lemma is a weaker form of a statement we will prove later in the lecture.

Lemma. Let A and B be $n \times n$ matrices with det $A \neq 0$. Then det $(AB) = (\det A)(\det B)$.

Proof. This property is not straightforward to show directly from the formula for det as a sum over permutation matrices. Instead, we take the following approach.

Define $f : \{ n \times n \text{ matrices } \} \to \mathbb{R}$ to be the function $f(M) = \frac{\det(AM)}{\det A}$.

This definition makes sense because det $A \neq 0$.

Key claim: then the function f has the three defining properties of the determinant. Therefore f must be equal to det since det is the **unique function with these properties**. In more detail:

- We have $f(I_n) = \frac{\det(AI_n)}{\det A} = \frac{\det A}{\det A} = 1.$
- If M' is given by swapping two columns in M, then AM' is given by swapping the two corresponding columns in AM, so $f(M') = \frac{\det(AM')}{\det A} = \frac{-\det(AM)}{\det A} = -f(M)$.
- If column *i* of *M* is *p* times column *i* of *M'* plus *q* times column *i* of *M''* and all other columns of *M*, *M'*, and *M''* are equal, then the same is true of *AM*, *AM'*, and *AM''* so

$$f(M) = \frac{\det(AM)}{\det A} = \frac{p\det(AM') + q\det(AM'')}{\det A} = pf(M') + qf(M'').$$

These properties uniquely characterize det, so f and det must be the same function.

Therefore $f(B) = \frac{\det(AB)}{\det A} = \det B$ for any $n \times n$ matrix B, so $\det(AB) = (\det A)(\det B)$.

3 Determinants of triangular and invertible matrices

An $n \times n$ matrix A is *upper-triangular* if all of its nonzero entries occur in positions on or above the diagonal positions $(1, 1), (2, 2), (3, 3), \ldots, (n, n)$. Such a matrix looks like

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

where the * entries can be any numbers. The zero matrix is considered to be upper-triangular.

An $n \times n$ matrix A is *lower-triangular* if all of its nonzero entries occur in positions on or below the diagonal positions. Such a matrix looks like

$$\left[\begin{array}{cccc} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{array}\right]$$

where the * entries can again be any numbers. The zero matrix is also considered to be lower-triangular.

The transpose of an upper-triangular matrix is lower-triangular, and vice versa.

We say that a matrix is *triangular* if it is either upper- or lower-triangular.

A matrix is *diagonal* if it is **both** upper- and lower-triangular.

This happens precisely when all nonzero entries are on the diagonal:

*	0	0	0
0	*	0	0
0	0	*	0
0	0	0	*

The *diagonal entries* of A are the numbers that occur in positions $(1, 1), (2, 2), (3, 3), \ldots, (n, n)$.

Proposition. If A is a triangular matrix then det A is the product of the diagonal entries of A.

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Proof. Assume A is upper-triangular.

If $X \in S_n$ and $X \neq I_n$ then at least one nonzero entry of X is in a position below the diagonal, in which case $\operatorname{prod}(X, A)$ is a product of numbers which includes 0 (since all positions below the diagonal in A contain zeros) and is therefore 0.

Hence det $A = \sum_{X \in S_n} \operatorname{prod}(X, A)(-1)^{\operatorname{inv}(X)} = \operatorname{prod}(I_n, A) =$ the product of the diagonal entries of A. If A is lower-triangular then the same result follows since det $A = \det(A^{\top})$.

Lemma. If A is an $n \times n$ matrix then det A is a **nonzero** multiple of det (RREF(A)).

Proof. Suppose B is obtained from A by an elementary row operation. To prove the lemma, it is enough to show that det B is a nonzero multiple of det A. There are three possibilities for B:

1. If B is formed by swapping two rows of A then B = XA for a permutation matrix $X \in S_n$.

Therefore det $B = \det(XA) = (\det X)(\det A) = \pm \det A.$

2. Suppose B is formed by rescaling a row of A by a nonzero scalar $\lambda \in \mathbb{R}$.

Then B = DA where D is a diagonal matrix of the form

$$D = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \lambda & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{bmatrix}$$

and in this case det $D = \lambda \neq 0$, so det $B = \det(DA) = (\det D)(\det A) = \lambda \det A$.

3. Suppose B is formed by adding a multiple of row i of A to row j.

Then B = TA for a triangular matrix T whose diagonal entries are all 1 and whose only other nonzero entry appears in column *i* and row *j*.

For example, if n = 4 and B is formed by adding 5 times row 2 of A to row 3 then

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

We therefore have $\det B = \det(TA) = (\det T)(\det A) = \det A$.

This shows that performing any elementary row operation to A multiplies det A by a nonzero number. It follows that det(RREF(A)) is a sequence of nonzero numbers times det A.

This brings us to an important property of the determinant that is worth remembering.

Theorem. An $n \times n$ matrix A is an invertible if and only if det $A \neq 0$.

Assume A is invertible. Then $\mathsf{RREF}(A) = I_n$, so $\det(\mathsf{RREF}(A)) = \det I_n = 1$.

Hence det $A \neq 0$ since det A is a nonzero multiple of det(RREF(A)).

Our next goal is to show that the determinant is a *multiplicative function*.

Lemma. Let A and B be $n \times n$ matrices. If A or B is not invertible then AB is not invertible.

Proof. Let X and Y be $n \times n$ matrices.

We have seen that X and Y are inverses of each other if $XY = I_n$, in which case also $YX = I_n$. Suppose AB is invertible with inverse X. Then $(AB)X = X(AB) = I_n$. Then A is invertible with $A^{-1} = BX$ since $A(BX) = (AB)X = I_n$. Likewise, B is invertible with $B^{-1} = XA$ since $(XA)B = X(AB) = I_n$. Thus, if A or B is not invertible then AB cannot be invertible.

Theorem. If A and B are any $n \times n$ matrices then det(AB) = (det A)(det B).

Proof. We already proved this in the case when det $A \neq 0$.

If det A = 0, then A is not invertible, so AB is not invertible either, so det $(AB) = 0 = (\det A)(\det B)$. \Box

It is difficult to derive this theorem directly from the formula det $A = \sum_{X \in S_n} \operatorname{prod}(X, A)(-1)^{\operatorname{inv}(X)}$.

Example. We have det $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 4 - 6 = -2$ and det $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = 10 - 12 = -2$. On the other hand, det $\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \right) = det \begin{bmatrix} 10 & 13 \\ 22 & 29 \end{bmatrix} = 290 - 286 = 4$.

4 Computing determinants

Our proof that det A is a nonzero multiple of $det(\mathsf{RREF}(A))$ can be turned into an effective algorithm.

Algorithm to compute det A (useful when A is larger than 3×3).

Input: an $n \times n$ matrix A.

- 1. Start by setting a scalar denom = 1.
- 2. Row reduce A to an echelon form E. It is not necessary to bring A all the way to reduced echelon form. We just need to row reduce A until we get an upper triangular matrix.

Each time you perform a row operation in this process, modify denom as follows:

- (a) When you switch two rows, multiply denom by -1.
- (b) When you multiply a row by a nonzero scalar λ , multiply denom by λ .
- (c) When you add a multiple of a row to another row, don't do anything to denom.

The determinant $\det E$ is the product of the diagonal entries of E.

The determinant of A is given by $\det A = \frac{\det E}{\operatorname{denom}}$.

Therefore

Example. We reduce the following matrix to echelon form:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 0 & -3 & -9 \\ 2 & 4 & 6 \end{bmatrix}$$
 denom = 1

$$\sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & -3 & -9 \\ 0 & -2 & -4 \end{bmatrix}$$
 (we added a multiple of row 1 to row 3) denom = 1

$$\sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & -2 & -4 \end{bmatrix}$$
 (we multiplied row 2 by -1/3) denom = -1/3

$$\sim \begin{bmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & -2 & -4 \end{bmatrix} = E$$
 (we added a multiple of row 2 to row 3) denom = -1/3
det $A = \frac{\det E}{\det onm} = \frac{1 \cdot 1 \cdot 2}{-1/3} = -6.$

Another algorithm to compute $\det A$ (useful when A has many entries equal to zero).

Define $A^{(i,j)}$ to be the submatrix formed by removing row *i* and column *j* from *A*.

For example, if $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ then $A^{(1,2)} = \begin{bmatrix} d & f \\ g & i \end{bmatrix}$.

Theorem. If A is the $n \times n$ matrix with entry a_{ij} row i and column j, then

(1)
$$\det A = a_{11} \det A^{(1,1)} - a_{12} \det A^{(1,2)} + a_{13} \det A^{(1,3)} - \dots - (-1)^n a_{1n} \det A^{(1,n)}$$

(2)
$$\det A = a_{11} \det A^{(1,1)} - a_{21} \det A^{(2,1)} + a_{31} \det A^{(3,1)} - \dots - (-1)^n a_{n1} \det A^{(n,1)}.$$

Note that each $A^{(1,j)}$ or $A^{(j,1)}$ is a square matrix smaller than A.

Thus det $A^{(1,j)}$ or det $A^{(j,1)}$ can be computed by the same formula on a smaller scale.

Proof. The second formula follows from the first formula since det $A = \det(A^{\top})$. (Why?)

The first formula is a consequence of the formula for $\det A$ we derived last lecture. One needs to show

$$-(-1)^{j}a_{1j}\det A^{(1,j)} = \sum_{X\in S_{n}^{(j)}}\operatorname{prod}(X,A)(-1)^{\operatorname{inv}(X)}$$

where $S_n^{(j)}$ is the set of $n \times n$ permutation matrices which have a 1 in column j of the first row. Summing the left expression over j = 1, 2, ..., n gives the desired formula.

Summing the right expression over j = 1, 2, ..., n gives $\sum_{X \in S_n} \operatorname{prod}(X, A)(-1)^{\operatorname{inv}(X)} = \det A$.

Example. This result can be used to derive our formula for the determinant of a 3-by-3 matrix:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix} = a(ei-fh) - b(di-fg) + c(dh-eg).$$

Vocabulary $\mathbf{5}$

Keywords from today's lecture:

1. Upper-triangular matrix.

A square matrix of the form $\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$ with zeros in all positions below the main diagonal.

2. Lower-triangular matrix.

A square matrix of the form $\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}$ with zeros in all positions above the main diagonal.

The transpose of an upper-triangular matrix.

3. Triangular matrix.

A matrix that is either upper-triangular or lower-triangular.

4. Diagonal matrix.

A square matrix of the form $\begin{vmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{vmatrix}$ with zeros in all non-diagonal pos	ix of the form	A square matrix of the form
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A matrix that is both upper-triangular and lower-triangular.