Instructions: Choose **4 problems** and write down detailed solutions, showing all necessary work. You can earn up to **4 extra credit points** by correctly solving additional problems.[1]

Some of the problems are more challenging than others, and there is no need to solve all of them. Problems that would not make reasonable exam questions are marked with a star.

To get full credit for the required part of the homework, you just need to make a good-faith attempt on 4 problems. The bar for receiving extra credit points for additional problems is higher.

You are free to discuss problems with other students and to consult whatever resources you want, but you must write up your own solutions. Solutions copied from somewhere else will receive zero credit.

Submission: Please handwrite your answers and show all steps in your calculations, as you would on an exam. **Submit your hard copy solutions** before the end of the day on the due date to your tutorial's homework submission box outside the 3rd floor math admin offices near Lift 25/26.

Please **coordinate with your tutorial TA directly** if you need to submit solutions electronically.

1. (a) For each $i = 1, 2, 3$, the map $T_i : \mathbb{R}^3 \to \mathbb{R}^2$ given by the formula

$$T_i(v) = (\text{the vector } v \text{ with row } i \text{ deleted})$$

   is a linear transformation. What are the standard matrices of T_1, T_2, and T_3?

   (b) Can you find two linearly independent vectors $v, w \in \mathbb{R}^3$ such that for each $i = 1, 2, 3$, if we delete row i from both vectors, we get linearly independent vectors in $\mathbb{R}^2$?

   In other words, such that the set $\{T_1(v), T_1(w)\}$ is linearly independent, $\{T_2(v), T_2(w)\}$ is linearly independent, and $\{T_3(v), T_3(w)\}$ is linearly independent. Justify your answer.

   (c) Can you find two linearly independent vectors $v, w \in \mathbb{R}^3$ such that for each $i = 1, 2, 3$, if we delete row i from both vectors, we get linearly dependent vectors in $\mathbb{R}^2$?

   In other words, such that the set $\{T_1(v), T_1(w)\}$ is linearly dependent, $\{T_2(v), T_2(w)\}$ is linearly dependent, and $\{T_3(v), T_3(w)\}$ is linearly dependent. Justify your answer.

*2. Suppose A is an $m \times n$ matrix and B is an $m \times q$ matrix.

   The *rank* of a matrix M, denote $\text{rank}(M)$, is its number of pivot columns.

   (a) What relationship, if any, exists between $\mathsf{RREF}(A)$ and $\mathsf{RREF}(\begin{bmatrix} A & B \end{bmatrix})$ in general?

   What relationship, if any, exists between $\text{rank}(A)$ and $\text{rank}(\begin{bmatrix} A & B \end{bmatrix})$ in general?

   (b) What relationship, if any, exists between $\mathsf{RREF}(B)$ and $\mathsf{RREF}(\begin{bmatrix} A & B \end{bmatrix})$ in general?

   What relationship, if any, exists between $\text{rank}(B)$ and $\text{rank}(\begin{bmatrix} A & B \end{bmatrix})$ in general?

   (c) What relationship, if any, exists between $\mathsf{RREF}(\begin{bmatrix} A & B \end{bmatrix})$ and $\mathsf{RREF}(\begin{bmatrix} B & A \end{bmatrix})$ in general?

   What relationship, if any, exists between $\text{rank}(\begin{bmatrix} A & B \end{bmatrix})$ and $\text{rank}(\begin{bmatrix} B & A \end{bmatrix})$ in general?

   For the first question in each part, the possible answers are

   "the two RREFs are equal" or
   "the smaller RREF is a specific submatrix of the larger RREF" or
   "no general relationship".

   For the second question in each part, the possible answers are: $\leq$, $\geq$, $=$, or no general relationship.

   Just pick from among these options, and provide some explanation for your choices.

[1] There will be ∼10 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼40 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

*3. Suppose A is an $m \times n$ matrix and C is a $q \times n$ matrix.

(a) What relationship, if any, exists between $\mathsf{RREF}(A)$ and $\mathsf{RREF}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ in general?

What relationship, if any, exists between $\mathrm{rank}(A)$ and $\mathrm{rank}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ in general?

(b) What relationship, if any, exists between $\mathsf{RREF}(C)$ and $\mathsf{RREF}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ in general?

What relationship, if any, exists between $\mathrm{rank}(C)$ and $\mathrm{rank}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ in general?

(c) What relationship, if any, exists between $\mathsf{RREF}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ and $\mathsf{RREF}\left(\begin{bmatrix} C \\ A \end{bmatrix}\right)$ in general?

What relationship, if any, exists between $\mathrm{rank}\left(\begin{bmatrix} A \\ C \end{bmatrix}\right)$ and $\mathrm{rank}\left(\begin{bmatrix} C \\ A \end{bmatrix}\right)$ in general?

Use the same set of possible answers as in the previous question.

4. Suppose $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ are vectors and $V = \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_k\}$.

(a) If we add another vector v_{k+1} to this list, will it still have the same span?

What can you say about when $V = \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_k, v_{k+1}\}$?

(b) If we delete one of the vectors from the list, say v_k, will it still have the same span?

What can you say about when $V = \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_{k-1}\}$?

Explain and justify your answers to both parts.

5. Let $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $w = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$.

The span $\mathbb{R}\text{-span}\{v, w\} = \{\alpha v + \beta w : \alpha, \beta \in \mathbb{R}\}$ is the smallest set of vectors in $\mathbb{R}^2$ containing v and w that is **closed under scalar multiplication and vector addition/subtraction**.

(a) Explain why $\mathbb{R}\text{-span}\{v, w\} = \mathbb{R}^2$.

(b) Draw a picture of the smallest set $\mathcal{S}$ of vectors in $\mathbb{R}^2$ that contains v and w and is closed under just scalar multiplication. This means that $v \in \mathcal{S}$, $w \in \mathcal{S}$, if $u \in \mathcal{S}$ then $cu \in \mathcal{S}$ for all $c \in \mathbb{R}$, and $\mathcal{S}$ is as small as possible with these properties.

(c) Draw a picture of the smallest set $\mathcal{T}$ of vectors in $\mathbb{R}^2$ that contains v and w and is closed under just vector addition and subtraction. This means that $v \in \mathcal{T}$, $w \in \mathcal{T}$, if $x, y \in \mathcal{T}$ then $x + y \in \mathcal{T}$ and $x - y \in \mathcal{T}$, and $\mathcal{T}$ is as small as possible with these properties.

(d) Draw pictures of $\frac{1}{2}\mathcal{S} = \{\frac{1}{2}u : u \in \mathcal{S}\}$ and $\frac{1}{2}\mathcal{T} = \{\frac{1}{2}u : u \in \mathcal{T}\}$.

In general, how is $\mathcal{S}$ related to $\frac{1}{n}\mathcal{S}$ and how is $\mathcal{T}$ related to $\frac{1}{n}\mathcal{T}$ when n is a positive integer?

*6. Suppose $v_1, v_2, \dots, v_k$ and $w_1, w_2, \dots, w_l$ are two lists of vectors in $\mathbb{R}^n$.

Describe an algorithm to determine whether or not $\mathbb{R}\text{-span}\{v_1, v_2, \dots, v_k\} = \mathbb{R}\text{-span}\{w_1, w_2, \dots, w_l\}$.

7. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$ is a linear transformation.

If $T\left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}\right) = T\left(\begin{bmatrix} 4 \\ 4 \\ 7 \end{bmatrix}\right)$ then what is $T\left(\begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix}\right)$?

Justify your answer.

8. Suppose $m \leq n$ are positive integers.

   Let X be a finite set with m elements and let Y be a finite set with n elements.

   Let $\mathbb{Z}$ be the infinite set of all integers.

   (a) Explain why any one-to-one map $f : Y \to X$ is also onto.

   Similarly, explain why any one-to-one linear map $f : \mathbb{R}^n \to \mathbb{R}^m$ is also onto.

   (b) Explain why any onto map $f : X \to Y$ is also one-to-one.

   Similarly, explain why any onto linear map $f : \mathbb{R}^m \to \mathbb{R}^n$ is also one-to-one.

   (c) These properties do not hold for arbitrary maps between infinite sets.

   To show this, give an example of a map $f : \mathbb{Z} \to \mathbb{Z}$ that is one-to-one but not onto.

   Then give an example of a map $g : \mathbb{Z} \to \mathbb{Z}$ that is onto but not one-to-one.

   (Suggestion: for parts (a) and (b) use the characterization of onto and one-to-one linear maps in terms of the reduced echelon form of the standard matrix.)

9. Consider the set of vectors $\mathcal{Q}$ in $\mathbb{R}^2$ whose endpoints are in the quadrilateral in the xy-plane with vertices $(0, 0)$, $(-20, 23)$, $(21, 21)$, and $(1, 44)$.

   If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation with $\left\{ T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) : 0 \leq a \leq 1 \text{ and } 0 \leq b \leq 1 \right\} = \mathcal{Q}$, then what are the possibilities for the standard matrix A of T? Justify your answer.

*10. The unit cube in $\mathbb{R}^n$ is the set $\mathcal{C} = \{v \in \mathbb{R}^n : 0 \leq v_i \leq 1 \text{ for all } i = 1, 2, \dots, n\}$.

   Find all $n \times n$ matrices A such that $A\mathcal{C} = \mathcal{C}$. Be sure to justify your answer.

   Here $A\mathcal{C}$ is defined to be the set $\{Av : v \in \mathcal{C}\}$.

11. Suppose $f : X \to X$ is a function with the same domain and codomain.

   For each a positive integer n, let $f^n : X \to X$ be the function with the formula

$$f^n(x) = \underbrace{f(f(f(\cdots f}_{n \text{ times}}(x)\cdots)))$$

   so that $f^1(x) = f(x)$, $f^2(x) = f(f(x))$, and $f^3(x) = f(f(f(x)))$, and so on.

   The function f has *finite order* if there is a positive integer n with $f^n = \mathrm{id}_X$.

   When f has finite order, its *order* is the smallest positive integer n with $f^n = \mathrm{id}_X$.

   How many functions $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ have finite order?

   What are the possible values for their orders? Justify your answer.

*12. An $n \times n$ matrix A has *finite order* if there is a positive integer n with $A^n = I$.

   When A has finite order, its *order* is the smallest positive integer n with $A^n = I$.

   (Notice that if $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear function with standard matrix A, then T has finite order if and only if A does, in which case the order of T is the same as the order of A.)

   Suppose A is a 2×2 matrix with **integer** entries that has finite order.

   What numbers n can occur as the order of A? Justify your answer.

   For each possible value of n, give an example of an integer 2×2 matrix A with order n.

*13. This problem has two options.

Either: ask ChatGPT or another LLM to explain a concept from this week's lecture that you found confusing. Print out a transcript of your conversation. You can only receive credit for this question if (1) the LLM's explanation is correct and (2) the explanation was genuinely helpful to your understanding. We will judge item (2) based on the length and depth of your transcript.

Or: find an instance where an LLM like ChatGPT gives an **incorrect explanation** when asked about a concept or problem related to this week's lecture. Print out a transcript of your conversation and then **explain what the error is**. You cannot receive credit for this question if the error is just a simple miscalculation or bad arithmetic. Try to encounter an interesting conceptual mistake.