Instructions: Choose **4 problems** and write down detailed solutions, showing all necessary work. You can earn up to **4 extra credit points** by correctly solving additional problems.[1]

Some of the problems are more challenging than others, and there is no need to solve all of them. Problems that would not make reasonable exam questions are marked with a star.

To get full credit for the required part of the homework, you just need to make a good-faith attempt on 4 problems. The bar for receiving extra credit points for additional problems is higher.

You are free to discuss problems with other students and to consult whatever resources you want, but you must write up your own solutions. Solutions copied from somewhere else will receive zero credit.

Submission: Please handwrite your answers and show all steps in your calculations, as you would on an exam. **Submit your hard copy solutions** before the end of the day on the due date to your tutorial's homework submission box outside the 3rd floor math admin offices near Lift 25/26.

Please **coordinate with your tutorial TA directly** if you need to submit solutions electronically.

1. Define the *two-sided reduced echelon form* of a matrix A to be

$$\mathsf{TREF}(A) := \mathsf{RREF}(\mathsf{RREF}(A)^\top)^\top.$$

Here $M^\top$ denotes the transpose of a matrix M.

As a warmup, calculate $\mathsf{TREF}(A)$ if

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix}.$$

Describe all the possibilities for $\mathsf{TREF}(A)$ when A is a general $m \times n$ matrix.

2. The *transpose* of a linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear function $g : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$f(v)^\top w = v^\top g(w) \quad \text{for all } v \in \mathbb{R}^n \text{ and } w \in \mathbb{R}^m.$$

Provide answers **with justification** to the following questions:

(a) If A is the standard matrix of f, then what is the standard matrix of the transpose of f?

(First try this for a specific matrix A like the one in problem #1. Then find a general solution.)

(b) Why is there only one transpose of f?

(In other words, suppose $h : \mathbb{R}^m \to \mathbb{R}^n$ is another transpose, and then explain why $g = h$.)

3. Find a general formula for A^n when $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and n is any positive integer.

(Hint: write $A = I + B$ where I the identity matrix. Then find a general formula for B^n first.)

4. Suppose $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R}$. When is

$$A = \begin{bmatrix} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & c_2 & 0 & d_2 \end{bmatrix}$$

invertible? Find a formula for A^{-1} when A is invertible.

[1] There will be ∼10 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼40 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

5. Suppose $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in \mathbb{R}$. When is

$$A = \begin{bmatrix} 0 & 0 & a_1 & b_1 \\ 0 & 0 & c_1 & d_1 \\ a_2 & b_2 & 0 & 0 \\ c_2 & d_2 & 0 & 0 \end{bmatrix}$$

invertible? Find a formula for A^{-1} when A is invertible.

6. Suppose $a, b \in \mathbb{R}$ and $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $D = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}$.

Find an invertible matrix P such that $AP = PD$.

Then find a general formula for A^n when n is any positive integer.

(Hint: first express A, A^2, and A^3 in terms of P and D.)

*7. Suppose $a, b, c, d \in \mathbb{R}$. Define $f : \mathbb{R} \to \mathbb{R}$ to be the function with formula

$$f(x) = ax^3 + bx^2 + cx + d.$$

Important note: we are defining f to have domain $\mathbb{R}$ and also codomain $\mathbb{R}$.

What are the possibilities for (a, b, c, d) if f is an invertible function $\mathbb{R} \to \mathbb{R}$?

(Hint: separately consider the cases when $a \neq 0$, $a = 0 \neq b$, $a = b = 0 \neq c$, etc., and use calculus.)

8. Suppose A is an $n \times n$ matrix and I is the $n \times n$ identity matrix.

Choose positive real numbers b and c.

(a) Assume $A^2 = bI + cA$. Find a formula for A^{-1}.

(b) Suppose A is the $n \times n$ matrix with b on the main diagonal and c everywhere else.

For example, if $n = 4$ then we would have

$$A = \begin{bmatrix} b & c & c & c \\ c & b & c & c \\ c & c & b & c \\ c & c & c & b \end{bmatrix}.$$

When is A invertible? Compute A^{-1} when A is invertible.

(Your answer should be for general $b, c > 0$ and general n, not just for $n = 4$.)

9. Suppose A and B are $n \times n$ matrices such that $v^\top A w = v^\top B w \in \mathbb{R}$ for all vectors $v, w \in \mathbb{R}^n$.

Does it always hold that A and B are equal? Prove that $A = B$ or find a counterexample.

10. The *commutator* of two invertible matrices A and B is $[A, B] := ABA^{-1}B^{-1}$.

Compute a formula for $[A, B]$ when $A = \begin{bmatrix} 1 & a & b & c \\ 0 & 1 & 0 & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & v & w & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Here $a, b, c, d, e, v, w, x, y, z$ are arbitrary real numbers.

11. Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$. Find all 3×2 matrices B such that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

12. Suppose A is an $m \times n$ matrix where $m \neq n$ and I is the $m \times m$ identity matrix.

Describe an algorithm to determine if there exists a $n \times m$ matrix B with $AB = I$, and to find such a matrix B when one does exist. Explain why your algorithm works.

13. Recall that the transpose of a matrix is formed by flipping the matrix across the northwest-to-southeast diagonal. The *anti-transpose* of a matrix is formed by flipping the matrix across the northeast-to-southwest diagonal.

    Denote the anti-transpose of a matrix A as $A^\dagger$. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}^\dagger = \begin{bmatrix} 7 & 3 \\ 6 & 2 \\ 5 & 1 \end{bmatrix} \quad \text{which is not equal to} \quad \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}^\top = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \end{bmatrix}.$$

    Explain why $(AB)^\dagger = B^\dagger A^\dagger$ whenever A and B are matrices such that AB is defined.

*14. This problem has two options.

    Either: ask ChatGPT or another LLM to explain a concept from this week's lecture that you found confusing. Print out a transcript of your conversation. You can only receive credit for this question if (1) the LLM's explanation is correct and (2) the explanation was genuinely helpful to your understanding. We will judge item (2) based on the length and depth of your transcript.

    Or: find an instance where an LLM like ChatGPT gives an **incorrect explanation** when asked about a concept or problem related to this week's lecture. Print out a transcript of your conversation and then **explain what the error is**. You cannot receive credit for this question if the error is just a simple miscalculation or bad arithmetic. Try to encounter an interesting conceptual mistake.