Instructions: Choose **4 problems** and write down detailed solutions, showing all necessary work. You can earn up to **4 extra credit points** by correctly solving additional problems.[1]

Some of the problems are more challenging than others, and there is no need to solve all of them. Problems that would not make reasonable exam questions are marked with a star.

To get full credit for the required part of the homework, you just need to make a good-faith attempt on 4 problems. The bar for receiving extra credit points for additional problems is higher.

You are free to discuss problems with other students and to consult whatever resources you want, but you must write up your own solutions. Solutions copied from somewhere else will receive zero credit.

Submission: Please handwrite your answers and show all steps in your calculations, as you would on an exam. **Submit your hard copy solutions** before the end of the day on the due date to your tutorial's homework submission box outside the 3rd floor math admin offices near Lift 25/26.

Please **coordinate with your tutorial TA directly** if you need to submit solutions electronically.

1. Let $A = \begin{bmatrix} 1 & 1 & 1 & 11 \\ 6 & 8 & 16 & -26 \\ 3 & 1 & -7 & 25 \end{bmatrix}$.

   (a) Find a basis for the column space of A.

   (b) Find a basis for the null space of A.

   Do all steps by hand and show your work.

2. Let $A = \begin{bmatrix} 2 & 3 & 16 & -5 \\ -6 & 8 & 20 & -2 \\ 3 & -2 & -8 & 3 \end{bmatrix}$.

   (a) Find a basis for the column space of A.

   (b) Find a basis for the null space of A.

   Do all steps by hand and show your work.

3. Let $A = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$ be a one-row matrix. Here $a_1, a_2, \dots, a_n \in \mathbb{R}$ are arbitrary numbers.

   (a) Find a basis for the column space of A.

   (b) Find a basis for the null space of A.

   Your answer will depend on the entries in A. Remember to the consider the case when $A = 0$.

4. Suppose $v_1, v_2, \dots, v_k$ are linearly independent vectors in $\mathbb{R}^n$, where $k < n$.

   Describe an algorithm to find vectors $v_{k+1}, v_{k+2}, \dots, v_n$ such that $v_1, v_2, \dots, v_n$ is a basis for $\mathbb{R}^n$.

*5. Show that the rank one $m \times n$ matrices (the $m \times n$ matrices A with $\mathrm{rank}(A) = \dim(\mathrm{Col}(A)) = 1$) are precisely the matrices that can be expressed as vw^T for vectors $0 \neq v \in \mathbb{R}^m$ and $0 \neq w \in \mathbb{R}^n$.

   One way to do this is by the following steps:

   (a) Explain why $\mathrm{rank}(vw^T) = 1$ if $0 \neq v \in \mathbb{R}^m$ and $0 \neq w \in \mathbb{R}^n$.

   (b) Explain why a rank one matrix A must have a nonzero column.

   (c) If v is a nonzero column of a rank one matrix A, explain how to find a nonzero vector w such that $A = vw^T$.

[1] There will be ∼10 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼40 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

6. Let $A = \begin{bmatrix} 2 & 0 & 4 & 0 \\ 1 & 8 & 0 & 5 \\ 2 & -8 & 4 & -4 \\ 0 & 0 & x & -9 \end{bmatrix}$.

   Determine the values of x such that A is invertible and find a formula for A^{-1} in this case.

*7. Suppose A is an $m \times n$ matrix and I_m is the $m \times m$ identity matrix.

   (a) Explain why there exists a matrix B with $AB = I_m$ if A has a pivot position in every row.

   (b) Prove that A has a pivot position in every row if there exists a matrix B such that $AB = I_m$

*8. Suppose A is an $m \times n$ matrix and I_n is the $n \times n$ identity matrix.

   (a) Explain why there exists a matrix B with $BA = I_n$ if A has a pivot position in every column.

   (b) Prove that A has a pivot position in every column if there exists a matrix B such that $BA = I_n$.

9. Suppose A is an $m \times n$ matrix and B is an $n \times q$ matrix. If $\mathrm{rank}(A) = n$ and $\mathrm{rank}(B) = r$, then what is $\mathrm{rank}(AB)$ in terms of m, n, q, and r? Justify your answer.

10. Let $v = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ and $w \in \mathbb{R}^3$. Suppose there exists a 3×3 matrix A whose null space **and** column space contains both v and w. What are the possibilities for w? For each of these possibilities, give an example of a 3×3 matrix A whose null space and column space contains both v and w.

*11. Let $\mathbb{R}^\infty$ be the set of vectors $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \end{bmatrix}$ with an infinite number of rows and all entries $v_i \in \mathbb{R}$.

   Vector addition and scalar multiplication for elements of $\mathbb{R}^\infty$ are defined as you would expect.

   The *span* of a set of vectors in $\mathbb{R}^\infty$ is the set of all of their *finite* linear combinations: even if your set has infinitely many elements, a linear combination can only involve finitely many of them (because there is no operation to add together an infinite set of nonzero vectors).

   For example, we can define the vectors $e_i \in \mathbb{R}^\infty$ as usual to have a 1 is row i and 0 in all other rows. The span of $e_1, e_2, e_3, \dots$ is *not* all of $\mathbb{R}^\infty$ because it does not contain the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$.

   Warmup: what subspace is the span of $e_1, e_2, e_3, \dots$ in $\mathbb{R}^\infty$? Describe this as concretely as you can.

   Then prove the following statement: if $b_1, b_2, b_3, \dots \in \mathbb{R}^\infty$ is any sequence of vectors indexed by the positive integers (in other words, a *countable* list of vectors), then $\mathbb{R}\text{-span}\{b_1, b_2, b_3, \dots\} \neq \mathbb{R}^\infty$.

12. Let A be an $n \times n$ matrix. The *generalized column space* of A is the set of vectors $v \in \mathbb{R}^n$ that are in $\mathrm{Col}(A^k)$ for **every** $k \in \{1, 2, 3, \dots\}$. Denote this subset by $\mathrm{Col}^+ A$.

   Similarly, the *generalized null space* of A is the set of vectors $v \in \mathbb{R}^n$ that are in $\mathrm{Nul}(A^k)$ for **at least one** $k \in \{1, 2, 3, \dots\}$. Denote this subset by $\mathrm{Nul}^+ A$.

   Show that $\mathrm{Nul}^+ A$ and $\mathrm{Col}^+ A$ are both subspaces of $\mathbb{R}^n$.

*13. Continue to assume A is an $n \times n$ matrix.

   Show that $v \mapsto Av$ defines an invertible function $\mathrm{Col}^+ A \to \mathrm{Col}^+ A$.

   Using this fact or some other method, show that $\mathrm{Nul}^+ A \cap \mathrm{Col}^+ A = \{0\}$, or in other words, that the only vector in $\mathbb{R}^n$ in both $\mathrm{Nul}^+ A$ and $\mathrm{Col}^+ A$ is the zero vector.

*14. Show that for each $u \in \mathbb{R}^n$ there are **unique** vectors $v \in \mathrm{Col}^+ A$ and $w \in \mathrm{Nul}^+ A$ with $u = v + w$.

(Use the previous exercise.)

15. The notation $\mathbb{R}[x_1, x_2, \dots, x_n]$ means the set of all polynomials in n commuting variables $x_1, x_2, \dots, x_n$ with coefficients in $\mathbb{R}$. Each $p \in \mathbb{R}[x_1, x_2, \dots, x_n]$ defines a function $\mathbb{R}^n \to \mathbb{R}$ by the formula

$$p\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}\right) = p(x_1, x_2, \dots, x_n).$$

So if $n = 3$ and $p = 2 + x_1 x_3^2 + x_2^3$ then $p\left(\begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}\right) = 2 + 3 \cdot 7^2 + 5^3 = 2 + 147 + 125 = 274.$

Assume $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are two linear functions. We already know how to form the functions $f + g : \mathbb{R}^n \to \mathbb{R}$ and $cf : \mathbb{R}^n \to \mathbb{R}$ for $c \in \mathbb{R}$.

In this situation there is one other operation we can use to produce a new function from f and g. This operation is called *pointwise multiplication* and is given as follows. Define $fg : \mathbb{R}^n \to \mathbb{R}$ to be the function with the formula $fg(x) = f(x)g(x)$ for $x \in \mathbb{R}^n$.

To make things more concrete below, suppose $n = 3$.

(a) Assume f and g have standard matrices

$$A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}.$$

As functions, f and g agree with two polynomials in $\mathbb{R}[x_1, x_2, x_3]$.

Warmup: what are these polynomials in terms of A and B?

(b) The pointwise product $fg : \mathbb{R}^3 \to \mathbb{R}$ is **not** a linear function, but it does agree with a polynomial in $\mathbb{R}[x_1, x_2, x_3]$. What is this polynomial in terms of A and B?

(c) Show that the smallest set of functions $\mathbb{R}^3 \to \mathbb{R}$ that contains all linear functions $\mathbb{R}^3 \to \mathbb{R}$ and is closed under addition, scalar multiplication, and pointwise multiplication is the subset of polynomials in $\mathbb{R}[x_1, x_2, x_3]$ **with no constant term**.

(In this context saying "S is the smallest subset with some property" means that S has the property and that if T is any other set with the property then $S \subseteq T$.)

Nothing here depends on $n = 3$, and we could reformulate these claims for any positive integer n.

16. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *homogeneous of degree d* if $f(cv) = c^d f(v)$ for all $v \in \mathbb{R}^n$.

Every linear function $\mathbb{R}^n \to \mathbb{R}$ is homogeneous of degree 1.

(a) Show that a function $f : \mathbb{R} \to \mathbb{R}$ that is homogeneous of degree 1 must be linear.

(b) Construct a function $f : \mathbb{R}^2 \to \mathbb{R}$ that is homogeneous of degree 1 but not linear.

(c) Explain why for general n the linear functions $\mathbb{R}^n \to \mathbb{R}$ are the same as the polynomials in $\mathbb{R}[x_1, x_2, \dots, x_n]$ that are homogeneous of degree 1.

17. This problem is about homogeneous polynomials of degree 2.

(a) Suppose $A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$ is an $n \times n$ matrix.

Explain why the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(v) = v^\top A v$ is homogeneous of degree two. Can you write this function as a polynomial in $\mathbb{R}[x_1, x_2, \dots, x_n]$?

(b) Show that if $f \in \mathbb{R}[x_1, x_2, \dots, x_n]$ is a homogeneous polynomial of degree 2 then there is a unique *symmetric* matrix $A = A^\top$ such that $f(v) = v^\top Av$ for all $v \in \mathbb{R}^n$.

Call this matrix the *standard matrix* of f.

(c) Let $f \in \mathbb{R}[x_1, x_2, \dots, x_n]$ be a homogeneous polynomial of degree 2 with standard matrix A. Also suppose $g : \mathbb{R}^k \to \mathbb{R}^n$ is a linear function with standard matrix B. Explain why $f \circ g$ is still a homogeneous polynomial of degree 2. What is its standard matrix?

*18. This problem has two options.

Either: ask ChatGPT or another LLM to explain a concept from this week's lecture that you found confusing. Print out a transcript of your conversation. You can only receive credit for this question if (1) the LLM's explanation is correct and (2) the explanation was genuinely helpful to your understanding. We will judge item (2) based on the length and depth of your transcript.

Or: find an instance where an LLM like ChatGPT gives an **incorrect explanation** when asked about a concept or problem related to this week's lecture. Print out a transcript of your conversation and then **explain what the error is**. You cannot receive credit for this question if the error is just a simple miscalculation or bad arithmetic. Try to encounter an interesting conceptual mistake.