Instructions: Choose **4 problems** and write down detailed solutions, showing all necessary work. You can earn up to **4 extra credit points** by correctly solving additional problems.[1]

Some of the problems are more challenging than others, and there is no need to solve all of them. Problems that would not make reasonable exam questions are marked with a star.

To get full credit for the required part of the homework, you just need to make a good-faith attempt on 4 problems. The bar for receiving extra credit points for additional problems is higher.

You are free to discuss problems with other students and to consult whatever resources you want, but you must write up your own solutions. Solutions copied from somewhere else will receive zero credit.

Submission: Please handwrite your answers and show all steps in your calculations, as you would on an exam. **Submit your hard copy solutions** before the end of the day on the due date to your tutorial's homework submission box outside the 3rd floor math admin offices near Lift 25/26.

Please **coordinate with your tutorial TA directly** if you need to submit solutions electronically.

1. Suppose $V = \{p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 : c_0, c_1, c_2, c_3 \in \mathbb{R}\}$ is the 4-dimensional vector space of polynomials of degree ≤ 3. Let $T : V \to V$ be the linear map defined by $T(p(x)) = p(1-x)$.

   This means that $T(1 + 2x + x^3) = 1 + 2(1-x) + (1-x)^3 = 4 - 5x + 3x^2 - x^3$, for example.

   Let $a_i = x^{i-1}$ and $b_i = (x+1)^{i-1}$ for $i = 1, 2, 3, 4$.

   Then a_1, a_2, a_3, a_4 and b_1, b_2, b_3, b_4 are two bases for V.

   As usual let e_1, e_2, e_3, e_4 be the standard basis of $\mathbb{R}^4$.

   There are invertible linear maps $f, g : \mathbb{R}^4 \to V$ with $f(e_i) = a_i$ and $g(e_i) = b_i$ for all $i = 1, 2, 3, 4$.

   This means that $f^{-1} \circ T \circ f$ and $g^{-1} \circ T \circ g$ and $f^{-1} \circ g$ are all linear maps $\mathbb{R}^4 \to \mathbb{R}^4$.

   Let A, B, and P be the standard matrices of $f^{-1} \circ T \circ f$, $g^{-1} \circ T \circ g$, and $f^{-1} \circ g$, respectively.

   Compute these matrices, and check that $\det(A) = \det(B)$ and $\mathrm{tr}(A) = \mathrm{tr}(B)$.

*2. Suppose V is an n-dimensional vector space and $T : V \to V$ is a linear map.

   Assume $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are two bases for V.

   As usual let $e_1, e_2, \dots, e_n$ be the standard basis of $\mathbb{R}^n$.

   There are invertible linear maps $f, g : \mathbb{R}^n \to V$ with $f(e_i) = a_i$ and $g(e_i) = b_i$ for all $i = 1, 2, \dots, n$.

   This means that $f^{-1} \circ T \circ f$ and $g^{-1} \circ T \circ g$ and $f^{-1} \circ g$ are all linear maps $\mathbb{R}^n \to \mathbb{R}^n$.

   Let A, B, and P be the standard matrices of $f^{-1} \circ T \circ f$, $g^{-1} \circ T \circ g$, and $f^{-1} \circ g$, respectively.

   Find an expression for A in terms of B and P, and use this to explain why A and B have the same trace and the same determinant.

   The determinant and trace of T are **defined** to be the values of $\det(A) = \det(B)$ and $\mathrm{tr}(A) = \mathrm{tr}(B)$. To compute these values, you have to pick a basis for V, but this exercise shows that the numbers you get are the same no matter which basis you use.

3. Adopt the same setup as in the previous problem.

   Explain why it still holds that T is invertible if and only $\det(T) \neq 0$.

   In other words, explain why T is an invertible linear map if and only if the matrix A is invertible.

[1] There will be ∼10 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼40 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

*4. For numbers $a, b, c, d \in \mathbb{R}$ define $\begin{bmatrix} a \\ c \end{bmatrix} \bullet \begin{bmatrix} b \\ d \end{bmatrix} = ab + cd$. For $v \in \mathbb{R}^2$ define $\|v\| = \sqrt{v \bullet v}$.

Suppose $v_1, v_2 \in \mathbb{R}^2$ are nonzero vectors. This exercise walks through a proof of the sometimes useful identity $v_1 \bullet v_2 = \|v_1\|\|v_2\| \cos(\theta)$ where θ is the angle between v_1 and v_2.

To define the angle θ precisely, assume that rotating v_1 counterclockwise by θ radians gives a **positive** scalar multiple of v_2, and that v_1 and v_2 are labeled such that this angle has $\theta \in [0, \pi]$. (This means that if we have to go more than π radians counterclockwise from v_1 to get to v_2, then we switch the names of the vectors.)

(a) Define u_1 and u_2 to be the unit vectors in the directions of v_1 and v_2, so $u_i = \frac{1}{\|v_i\|} v_i$.

Explain why $v_1 \bullet v_2 = \|v_1\|\|v_2\|(u_1 \bullet u_2)$.

(b) Suppose $\psi \in [0, 2\pi)$ and $M = \begin{bmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{bmatrix}$. Check that $(Mu_1) \bullet (Mu_2) = u_1 \bullet u_2$.

(c) Explain why you can choose a value of ψ such that $Mu_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $Mu_2 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ where θ is the angle between v_1 and v_2. Conclude that $u_1 \bullet u_2 = \cos(\theta)$.

Combining all three parts tells us that $v_1 \bullet v_2 = \|v_1\|\|v_2\| \cos(\theta)$.

5. Do there exist two linearly independent vectors in $\mathbb{R}^4$ that are orthogonal to all three of the vectors

$$\begin{bmatrix} 1 \\ -2 \\ 1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 7 \\ 1 \\ 4 \\ 7 \end{bmatrix}?$$

Find two such vectors if they exist, and otherwise explain why there are no such vectors.

6. Consider the plane $P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : 3x - 2y + 6z = 0 \right\}$ in $\mathbb{R}^3$.

(a) The subspace P is 2-dimensional. Find an orthogonal basis for P.

(b) Find the vector in P that is closest to $v = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

7. Find an orthonormal basis for the subspace of vectors of the form

$$\begin{bmatrix} a + 3b + 2c \\ 3a + 2b + 4c \\ 2a + 5b + 4c \\ 6a + 5b + 4c \end{bmatrix}$$

where $a, b, c \in \mathbb{R}$ are real numbers.

*8. Suppose A is a square matrix with all entries in $\mathbb{R}$.

Let $a, b \in \mathbb{R}$ and suppose $\lambda = a + bi \in \mathbb{C}$ is an eigenvalue for A.

(a) Show that if $A^\top = -A$ then $a = 0$.

(b) Show that if $A^\top = A$ then $b = 0$.

(c) Show that if $A^\top = A^{-1}$ then $a^2 + b^2 = 1$.

*9. Suppose A is a square matrix with all entries in $\mathbb{C}$. Determine whether the eigenvalue properties in the previous exercise still hold. That is, for each of the three statements, either find a counterexample or show that the given property for λ is still true when A has complex entries.

*10. Define

$$\mathbf{1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad j = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad k = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

As with complex numbers, when $a, b, c, d \in \mathbb{R}$ we abbreviate by setting

$$a + bi + cj + dk = a\mathbf{1} + bi + cj + dk = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}.$$

Now consider the real vector space of *quaternionic numbers* $\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}$.

(a) Compute all nine products yz for $y, z \in \{i, j, k\}$.

Conclude that $\mathbb{H}$ is closed under multiplication, that is, if $y, z \in \mathbb{H}$ then $yz \in \mathbb{H}$.

However, multiplication in $\mathbb{H}$ is not commutative (as it is in $\mathbb{R}$ and $\mathbb{C}$).

Why is multiplication in $\mathbb{H}$ associative? (In the sense that $x(yz) = (xy)z$ for all $x, y, z \in \mathbb{H}$.)

(b) Suppose $z = a + bi + cj + dk \in \mathbb{H}$. Find a formula for $\det(z)$.

(c) Suppose $z = a + bi + cj + dk \in \mathbb{H}$ is nonzero.

Find a formula for z^{-1} and check that this element is also still in $\mathbb{H}$.

*11. Suppose $p(z) = a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0$ is polynomial with $n > 0$, $a_0, \ldots, a_n \in \mathbb{C}$, and $a_n \neq 0$.

Two different versions of the fundamental theorem of algebra:

(A) There are complex numbers $r_1, r_2, \ldots, r_n \in \mathbb{C}$ such that $p(z) = a_n(z - r_1)(z - r_2) \cdots (z - r_n)$.

(B) There is a complex number $r \in \mathbb{C}$ with $p(r) = 0$.

The goal of this exercise is the prove these two statements are equivalent.

More precisely, we want to see that if (A) holds for any polynomial $p(z)$ of the given form, then (B) also holds for any polynomial $p(z)$ of the given form, and if (B) holds for any polynomial $p(z)$ of the given form, then (A) also holds for any polynomial $p(z)$ of the given form.

(a) Warmup: assume (A) holds for $p(z)$. Explain why (B) holds for $p(z)$.

This shows that (A) implies (B). The next two parts show that (B) implies (A).

(b) Suppose $r \in \mathbb{C}$ satisfies $p(r) = 0$.

Explain why there are complex numbers $b_0, \ldots, b_{n-1} \in \mathbb{C}$ with $b_{n-1} \neq 0$ such that

$$p(z + r) = (b_{n-1} z^{n-1} + \cdots + b_2 z^2 + b_1 z + b_0) z. \qquad (*)$$

How is b_{n-1} related to a_n?

(c) Continue the setup of part (b).

Use (*) to explain why there are complex numbers $c_0, \ldots, c_{n-1} \in \mathbb{C}$ with $c_{n-1} \neq 0$ such that

$$p(z) = g(z)(z - r) \quad \text{if we define} \quad g(z) = c_{n-1} z^{n-1} + \cdots + c_2 z^2 + c_1 z + c_0.$$

How is c_{n-1} related to b_{n-1}? How is c_0 related to b_0?

Now assume (B) holds. We deduce (A) using an argument by contradiction: suppose $p(z)$ is a polynomial of minimal degree $n > 0$ for which there is no factorization of the form in (A).

Property (B) tells us that there is some $r \in \mathbb{C}$ with $p(r) = 0$.

This means by (b) and (c) that $p(z) = g(z)(z - r)$ can be partially factored.

However, $g(z)$ has smaller degree $n - 1$ than $p(z)$, so by hypothesis $g(z)$ has a factorization of the form in (A), which means that $p(z) = g(z)(z - r)$ also has such a factorization $\rightsquigarrow$ contradiction.

*12. Suppose $p(z) = a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0$ is polynomial with $n > 0$, $a_0, \ldots, a_n \in \mathbb{C}$, and $a_n \neq 0$.

This exercise walks through a proof of the claim that

- There is a complex number $r \in \mathbb{C}$ with $p(r) = 0$.

In the previous exercise we saw that this is equivalent to the *Fundamental Theorem of Algebra*.

This exercise is fairly long, but if you want to understand why this fundamental theorem is true, it may be interesting!

(a) Warmup: what number works for r if we have $a_0 = 0$?

For the rest of this exercise assume $0 \neq a_0 \in \mathbb{C}$.

For $0 < N \in \mathbb{R}$ define $C(N) \subseteq \mathbb{R}^2$ to be the set of points (x, y) where x is the real part of

$$p(N \cdot \cos\theta + N \cdot \sin\theta \cdot i)$$

and y is the imaginary part, as $\theta \in \mathbb{R}$ varies from 0 to 2π.

$C(N)$ is a closed curve in xy-plane that starts and ends at the point $(x, y) = (\mathsf{Re}(p(N)), \mathsf{Im}(p(N)))$.

Define $W(N)$ to be the number of times $C(N)$ travels counter-clockwise completely around the origin $(x, y) = (0, 0)$ as θ varies from 0 to 2π. This called the *winding number* of $p(z)$.

(b) Explain why for any positive integer n we have

$$(N \cdot \cos\theta + N \cdot \sin\theta \cdot i)^n = N^n \cdot \cos(n\theta) + N^n \cdot \sin(n\theta) \cdot i.$$

Use the fact that $N \cdot \cos\theta + N \cdot \sin\theta \cdot i$ is N times a 2×2 rotation matrix.

(c) Suppose $p(z) = z^n + 1$.

Deduce that $C(N)$ draws a circle of radius N^n centered at $(x, y) = (1, 0)$, which travels n times around its center point as θ varies from 0 to 1.

In this case, what is $W(N)$ if $0 < N < 1$ and what is $W(N)$ if $N > 1$?

Draw a picture of $C(N)$ for each of these two cases.

(d) Suppose $p(z) = z^4 - 4z^3 + 5$.

Find a way to draw $C(N)$ and compute $W(N)$ where $N = 0.1$, 1, 2, 10, and 100.

One way to do this is to use the *parametric plot* feature of Wolfram Alpha.

Do some experiments to convince yourself that when $N \approx 0$ is very small the curve $C(N)$ is always very close to the nonzero point $(x, y) = (5, 0)$ so $W(N) = 0$, while if $N \gg 0$ is very large then $C(N)$ looks like a circle of radius N^4 that travels 4 times around the origin so $W(N) = 4$.

Moreover, as N increases, the value of $W(N)$ can only change if there is real number $N > 0$ such that the curve $C(N)$ passes through the origin.

(e) Suppose for a general $p(z)$ the curve $C(N)$ contains the origin $(0, 0)$.

Explain why this means there is some $r \in \mathbb{C}$ with $p(r) = 0$.

The observations you made in (b) for the given polynomial $p(z)$ are completely generic.

For any $p(z)$, when $N \approx 0$ is very small the curve $C(N)$ is very close to the nonzero point

$$(x, y) = (\mathsf{Re}(a_0), \mathsf{Im}(a_0))$$

so $W(N) = 0$, while if $N \gg 0$ is very large then $C(N)$ looks like a circle of radius N^n centered at the same point, which travels n times around the origin so $W(N) = n$.

But in order for $W(N)$ to change value, there must be some $N > 0$ such that $(0, 0) \in C(N)$. Then, using (e), we deduce that there is some $r \in \mathbb{C}$ with $p(r) = 0$.

*13. This problem has two options.

Either: ask ChatGPT or another LLM to explain a concept from this week's lecture that you found confusing. Print out a transcript of your conversation. You can only receive credit for this question if (1) the LLM's explanation is correct and (2) the explanation was genuinely helpful to your understanding. We will judge item (2) based on the length and depth of your transcript.

Or: find an instance where an LLM like ChatGPT gives an **incorrect explanation** when asked about a concept or problem related to this week's lecture. Print out a transcript of your conversation and then **explain what the error is**. You cannot receive credit for this question if the error is just a simple miscalculation or bad arithmetic. Try to encounter an interesting conceptual mistake.