Instructions: Choose **4 problems** and write down detailed solutions, showing all necessary work. You can earn up to **4 extra credit points** by correctly solving additional problems.[1]

Some of the problems are more challenging than others, and there is no need to solve all of them. Problems that would not make reasonable exam questions are marked with a star.

To get full credit for the required part of the homework, you just need to make a good-faith attempt on 4 problems. The bar for receiving extra credit points for additional problems is higher.

You are free to discuss problems with other students and to consult whatever resources you want, but you must write up your own solutions. Solutions copied from somewhere else will receive zero credit.

Submission: Please handwrite your answers and show all steps in your calculations, as you would on an exam. **Submit your hard copy solutions** before the end of the day on the due date to your tutorial's homework submission box outside the 3rd floor math admin offices near Lift 25/26.

Please **coordinate with your tutorial TA directly** if you need to submit solutions electronically.

1. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 6 \\ 5 \\ 5 \end{bmatrix}$.

   Find the orthogonal projection of b onto the orthogonal complement of the column space of A.

   Show all the steps in your derivation to receive credit.

2. Find all least-squares solutions to the linear equation $Ax = b$ where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}.$$

   Show all the steps in your derivation to receive credit.

3. Suppose $A = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 2 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 1 \\ h \end{bmatrix}$.

   (a) For which values of $h \in \mathbb{R}$ does $Ax = b$ have an exact solution?

   For such h, find all exact solutions to $Ax = b$.

   (b) For which values of $h \in \mathbb{R}$ does $Ax = b$ have a least-squares solution?

   For such h, find all least-squares solutions to $Ax = b$.

   Show all the steps in your derivation to receive credit.

4. Suppose $b \in \mathbb{R}^n$ is a list of numeric data.

   There exists a matrix A such that the *mean* of b is the unique least-squares solution $\widehat{x}$ to $Ax = b$ and the *(uncorrected sample) standard deviation* of b is $\frac{1}{\sqrt{n}}\|A\widehat{x} - b\|$.

   What is A?

   Explain your answer (which should depend on n but not on b).

[1] There will be ∼10 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼40 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

5. Suppose $A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix}$.

   Find an orthogonal matrix U and a diagonal matrix D such that $A = UDU^\top$.

   Show all the steps in your derivation to receive credit.

6. Suppose $A = A^\top$ is a symmetric $n \times n$ matrix with all real entries.

   (a) Explain why $v^\top Av > 0$ for all nonzero $v \in \mathbb{R}^n$ if A has all positive eigenvalues.

   (b) Explain why $v^\top Av = 0$ for some nonzero $v \in \mathbb{R}^n$ if A has zero as an eigenvalue.

   (c) Explain why $v^\top Av < 0$ for some $v \in \mathbb{R}^n$ if A has a negative eigenvalue.

   This shows that $v^\top Av > 0$ for all nonzero $v \in \mathbb{R}^n$ if and only if A has all positive eigenvalues.

   A symmetric matrix with this property is called *positive definite*.

   This also shows that $v^\top Av \geq 0$ for all $v \in \mathbb{R}^n$ if and only if A has all nonnegative eigenvalues.

   A symmetric matrix with this property is called *positive semidefinite*.

7. Suppose $A = A^\top$ is a symmetric $n \times n$ matrix with all real entries.

   Explain why $A = B^\top B$ for some $n \times n$ matrix B (possibly with complex entries).

   Compute a value for B when $A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix}$.

8. Suppose $A = A^\top$ is a positive semidefinite symmetric $n \times n$ matrix with all real entries.

   Explain how to find a positive semidefinite symmetric $n \times n$ matrix B with all real entries such that

   $$A = B^2.$$

   Compute B when $A = \begin{bmatrix} 40 & -28 & -26 \\ -28 & 52 & 2 \\ -26 & 2 & 25 \end{bmatrix}$.

9. Find the values of $x, y \in \mathbb{R}$ that minimize the distance between the vectors $\begin{bmatrix} x \\ x \\ x \end{bmatrix}$ and $\begin{bmatrix} y \\ 3y \\ -1 \end{bmatrix}$.

   Show all the steps in your derivation to receive credit.

*10. Suppose $a_1 < a_2 < a_3$ are real numbers and $b_1, b_2, b_3 \in \mathbb{R}$.

   Explain why the points $(x, y) = (a_1, b_1), (a_2, b_2), (a_3, b_3)$ are all on the same line if and only if

   $$a_1(b_3 - b_2) + a_2(b_1 - b_3) + a_3(b_2 - b_1) = 0.$$

   (One approach: consider the line of best fit through the three points.)

*11. Suppose A is a 2×2 matrix.

   (a) Can you always find an invertible matrix B such that $BAB^{-1} = A^\top$?

   (b) Can you always find an invertible symmetric matrix $C = C^\top$ such that $CAC^{-1} = A^\top$?

   In each case, **explain how to construct** B **and** C (if they exist).

12. Suppose A is an $n \times n$ matrix. **Assume** A **is diagonalizable.** Explain why $A^\top$ is similar to A.

*13. This problem has two options.

Either: ask ChatGPT or another LLM to explain a concept from this week's lecture that you found confusing. Print out a transcript of your conversation. You can only receive credit for this question if (1) the LLM's explanation is correct and (2) the explanation was genuinely helpful to your understanding. We will judge item (2) based on the length and depth of your transcript.

Or: find an instance where an LLM like ChatGPT gives an **incorrect explanation** when asked about a concept or problem related to this week's lecture. Print out a transcript of your conversation and then **explain what the error is**. You cannot receive credit for this question if the error is just a simple miscalculation or bad arithmetic. Try to encounter an interesting conceptual mistake.