<u>Instructions</u>: Choose **4 problems** and write down detailed solutions, showing all necessary work. You can earn up to **4 extra credit points** by correctly solving additional problems.[1]

Some of the problems are more challenging than others, and there is no need to solve all of them. Problems that would not make reasonable exam questions are marked with a star.

To get full credit for the required part of the homework, you just need to make a good-faith attempt on 4 problems. The bar for receiving extra credit points for additional problems is higher.

You are free to discuss problems with other students and to consult whatever resources you want, but you must write up your own solutions. Solutions copied from somewhere else will receive zero credit.

<u>Submission</u>: Please handwrite your answers and show all steps in your calculations, as you would on an exam. **Submit your hard copy solutions** before the end of the day on the due date to your tutorial's homework submission box outside the 3rd floor math admin offices near Lift 25/26.

Please **coordinate with your tutorial TA directly** if you need to submit solutions electronically.

1. Find an orthogonal matrix U and a diagonal matrix D such that

$$A = UDU^T \quad \text{for the matrix } A = \begin{bmatrix} 2 & 4 & 4 \\ 4 & 8 & 8 \\ 4 & 8 & 8 \end{bmatrix}.$$

2. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 3 & 0 \end{bmatrix}$$

   Show all steps in your derivation (but feel free to check your answer with a calculator).

3. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}.$$

   Show all steps in your derivation (but feel free to check your answer with a calculator).

4. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \\ 4 & 0 \\ 0 & 1 \end{bmatrix}.$$

   Show all steps in your derivation (but feel free to check your answer with a calculator).

5. Find a singular value decomposition for the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

   Show all steps in your derivation (but feel free to check your answer with a calculator).

[1] There will be ∼10 weeks of assignments, each with ∼10 practice problems, so you can earn up to ∼40 equally weighted extra credit points. The maximum amount of extra credit you can earn is 5% of your total grade for the semester.

6. Suppose A is a 2×2 matrix with a singular value decomposition
$$A = U\Sigma V^T$$
where U and V are orthogonal 2×2 matrices and
$$\Sigma = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}.$$
The first column of U is the vector $\begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$.

Draw a picture of the region in $\mathbb{R}^2$ given by
$$\left\{ Ax : x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \text{ is a vector with } x_1^2 + x_2^2 \leq 1 \right\}.$$

7. Suppose $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ are orthonormal vectors. Assume $k < n$.
   (a) Describe an algorithm to find $n - k$ vectors
   $$v_{k+1}, v_{k+2}, \dots, v_n \in \mathbb{R}^n$$
   such that $v_1, v_2, \dots, v_n$ is an orthonormal basis for $\mathbb{R}^n$.
   (b) Suppose $n = 3$ and $v_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$.
   Find two vectors $v_2, v_3 \in \mathbb{R}^3$ such that v_1, v_2, v_3 is an orthonormal basis for $\mathbb{R}^n$.

*8. Suppose A is an $m \times n$ matrix with at most one nonzero entry in each row and column.

Describe a singular value decomposition for A.

Use the following notation in your answer: suppose the positions of A with nonzero entries are
$$(i_1, j_1), (i_2, j_2), \dots, (i_r, j_r)$$
where $j_1 < j_2 < \cdots < j_r$, and the entries in these positions are $a_1, a_2, \dots, a_r \in \mathbb{R}$.

(It may be useful to compare your answer with #2 and #5.)

*9. Suppose A is an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$.

Suppose σ_k is much larger than σ_{k+1} where $k < \mathrm{rank}(A)$.

Describe an algorithm using an SVD for A that produces a good rank k approximation to A.

Apply this algorithm (using a computer or calculator) to find a rank 2 approximation to the matrix
$$A = \begin{bmatrix} 0.2 & 0.1 & -0.1 \\ 1.2 & 0.1 & 0.8 \\ 1.0 & -2.0 & 5.5 \end{bmatrix}.$$

*10. Suppose A is an invertible 3×3 matrix with a singular value decomposition
$$A = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}^\top.$$
Give a geometric interpretation of the numbers $\sigma_1, \sigma_2, \sigma_3$ and the vectors $u_1, u_2, u_3, v_1, v_2, v_3 \in \mathbb{R}^3$.

Your answer should involve the sphere $\mathbb{S} = \{w \in \mathbb{R}^3 : \|w\| \leq 1\}$ and its image $A\mathbb{S} = \{Aw : w \in \mathbb{S}\}$.

11. Give your favorate proof that $\text{rank}(A) = \text{rank}(A^\top)$ for any $m \times n$ matrix A.

12. Let A be a square $n \times n$ matrix whose rows are orthonormal.

    Explain why the columns of A are orthonormal.

*13. Let A be an $m \times n$ matrix. Describe an algorithm to find a unit vector $v \in \mathbb{R}^n$ such that

$$\|Av\| = \max\{\|Ax\| : x \in \mathbb{R}^n \text{ with } \|x\| = 1\}.$$

*14. Prove that any square matrix A is similar to its transpose $A^\top$.

*15. This problem has two options.

Either: ask ChatGPT or another LLM to explain a concept from this week's lecture that you found confusing. Print out a transcript of your conversation. You can only receive credit for this question if (1) the LLM's explanation is correct and (2) the explanation was genuinely helpful to your understanding. We will judge item (2) based on the length and depth of your transcript.

Or: find an instance where an LLM like ChatGPT gives an **incorrect explanation** when asked about a concept or problem related to this week's lecture. Print out a transcript of your conversation and then **explain what the error is**. You cannot receive credit for this question if the error is just a simple miscalculation or bad arithmetic. Try to encounter an interesting conceptual mistake.