MATH 2121 — Lecture #2

1. Review
2. Echelon form + Reduced echelon form
   a. How to use to solve linear systems
   b. How to compute these forms

Announcements:
1. confirmed midterm date: 17 Oct 8-10pm
2. will lecture recording

Ex $\begin{cases} x_1 + x_2 = 2 \\ x_1 - x_2 = 0 \end{cases}$

1 Last time: linear systems and row operations

Here's what we did last time: a *system of linear equations* or *linear system* is a list

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
$$

where $x_1, x_2, \dots, x_n$ are variables and each a_{ij} and b_i is a number.

The *coefficient matrix* and *augmented matrix* of such a system are respectively

$$
\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad \left[\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right].
$$

The coefficient matrix is $m \times n$. The augmented matrix is $m \times (n+1)$.

coeff matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ augmented matrix $\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right]$

$\begin{cases} x_1 + x_2 = 2 \\ x_1 - x_2 = 0 \end{cases}$ equiv to $\begin{cases} x_1 = 1 \\ x_2 = 1 \end{cases}$ equiv to $\begin{cases} x_1 + x_2 = 2 \\ 3x_1 + x_2 = 4 \end{cases}$ etc.

only has one solution $(x_1, x_2) = (1, 1)$

A *solution* to a linear system is a list of numbers $(s_1, s_2, \ldots, s_n)$ such that setting $x_1 = s_1$, $x_2 = s_2$, $\ldots$, $x_n = s_n$, all at the same time, makes each equation in the system a true statement.

Two linear systems are *equivalent* if they have the same solutions.

Important fact: Any linear system has either 0, 1, or infinitely many solutions.

We solve a linear system by performing *row operations* on its augmented matrix.

The following are row operations:

1. Replace one row by the sum of itself and a multiple of another row.
2. Multiply all entries in one row by a fixed nonzero number.
3. Interchange two rows.

Let's do an example to see these rules in action.

Examples of row operations:

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 0 & 6 \\ 2 & 2 & 2 & 2 \\ 4 & 5 & 0 & 0 \end{bmatrix} \xrightarrow[\text{(by factor } -\frac{1}{2})]{\text{② rescale}} \begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 0 & 6 \\ -1 & -1 & -1 & -1 \\ 4 & 5 & 0 & 0 \end{bmatrix}$$

aug. matrix of ...

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 5 \\ x_2 = 6 \\ 2x_1 + 2x_2 + 2x_3 = 2 \\ 4x_1 + 5x_2 = 0 \end{cases}$$

① replacement

replaced row 3 by 10 times row 2 + row 3

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 0 & 1 & 0 & 6 \\ 2 & 12 & 2 & 62 \\ 4 & 5 & 0 & 0 \end{bmatrix}$$

③ swap rows (rows 2 & 4)

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 4 & 5 & 0 & 0 \\ 2 & 2 & 2 & 2 \\ 0 & 1 & 0 & 6 \end{bmatrix}$$

<u>Thm</u> Row ops on aug. matrix of linear system produce aug. matrix of equivalent system

Example. Consider the linear system

$$1\quad\left\{\begin{aligned} x_1 + 2x_2 + 5x_3 &= 1 \\ x_1 + x_3 &= 0 \\ x_2 + x_3 &= 7 \end{aligned}\right. \quad \text{which has augmented matrix} \quad \begin{bmatrix} 1 & 2 & 5 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix}$$

Adding -1 times the second row to the first is an example of row operation (1):

$$-1 \cdot [1\ 0\ 1\ 0] = [-1\ 0\ -1\ 0] \qquad \begin{bmatrix} 1 & 2 & 5 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 0 & 2 & 4 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix}.$$

Next let's add -2 times the last row to the first row:

$$\begin{bmatrix} 0 & 2 & 4 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 0 & 0 & 2 & -13 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix}.$$

$$-2 \cdot [0\ 1\ 1\ 7] = [0\ -2\ -2\ -14]$$

Now lets use rule (3) to swap some rows:

$$\begin{bmatrix} 0 & 0 & 2 & -13 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} 0 & 1 & 1 & 7 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -13 \end{bmatrix} \xrightarrow{(3)} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 2 & -13 \end{bmatrix}.$$

Now lets scale the third row by 1/2:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 2 & -13 \end{bmatrix} \xrightarrow{(2)} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & -13/2 \end{bmatrix}.$$

Finally, lets use (1) twice to cancel the entries in rows 1 and 2 in column 3:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & -13/2 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{13/2} \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 1 & -13/2 \end{bmatrix} \xrightarrow{(1)} \begin{bmatrix} 1 & 0 & 0 & 13/2 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{27/2} \\ 0 & 0 & 1 & -13/2 \end{bmatrix}.$$

aug. matrix of the linear system

Two linear systems are *row equivalent* if their augmented matrices can be transformed to each other by a sequence of zero or more row operations.

Systems 1 + 2 are row equivalent

$$2 \quad \begin{cases} x_1 = 13/2 \\ x_2 = 27/2 \\ x_3 = -13/2 \end{cases}$$

Theorem. Row equivalent linear systems are equivalent, which means they have the same solutions.

Therefore the original system in our example has the same solutions as the system corresponding to the last matrix, which consists of the three equations

$$\begin{cases} x_1 = 13/2 \\ x_2 = 27/2 \\ x_3 = -13/2. \end{cases}$$

This system has only one solution $(13/2, 27/2, -13/2)$, so the original system also has only one solution.

2 Row reduction to echelon form

The goal today is to give an algorithm to determine whether a linear system has 0, 1, or infinitely many solutions, and to find out what these solutions are when they exist.

The algorithm will be called *row reduction to echelon form* and will formalize the way we solved the linear system in the last example. Sometimes, this algorithm is also called *Gaussian elimination.*

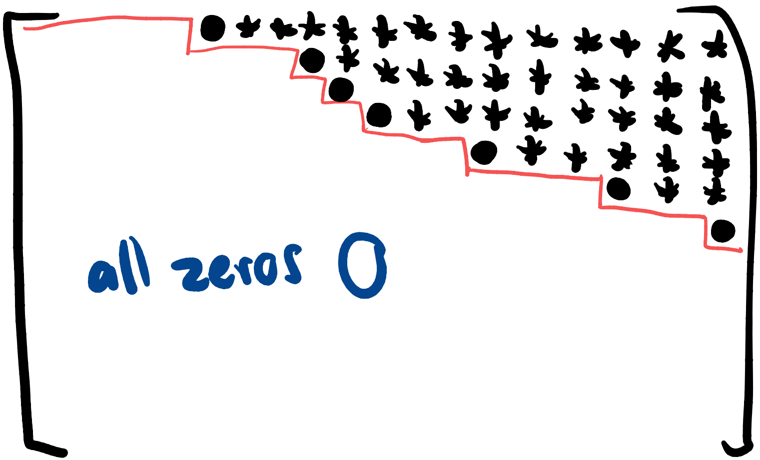
Goals: define echelon form and reduced echelon form

picture of [a matrix in echelon form]

blank space = zeros

● = nonzero entry

* = any entry



generic picture

Concrete example:

$$\begin{bmatrix} 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

zero row looks like $[0\ 0\ 0\ 0 \cdots 0]$

2.1 Defining reduced echelon form

A row in a matrix $\begin{bmatrix} a & b & \dots & z \end{bmatrix}$ is *nonzero* if not every entry in the row is zero.

A *nonzero column* in a matrix is defined similarly.

The *leading entry* in a row of a matrix is the first nonzero entry (from left to right).

For example, $\begin{bmatrix} 0 & 0 & \boxed{7} & 0 & 5 \end{bmatrix}$ has leading entry 7.

The leading entry occurs in column 3.

when a row is zero, every row below is also zero

Definition. A matrix with m rows and n columns is in $\boxed{\textit{echelon form}}$ if both:

(E1) When a row is nonzero, every row above it is also nonzero.

(E2) The leading entry in a nonzero row is strictly to the right of the leading entry of any earlier row.

The second property implies this additional property of a matrix in echelon form:

(E3) If a row is nonzero, then every entry below its leading entry in the same column is zero.

Some examples are helpful to understand this definition.

$$\begin{bmatrix} 0 & 0 & 1 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 2 & \cdots \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 0 & 0 & 0 & 4 & \cdots \\ & & & 0 & \\ & & & 0 & \\ & & & 0 & \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 2 & \cdots \\ 0 & 0 & 1 & \cdots & \cdots \end{bmatrix} \times \qquad \begin{bmatrix} 0 & 0 & 0 & 3 & \cdots \\ 0 & 0 & 0 & 5 & \cdots \end{bmatrix} \times$$

3×2 matrices in $\boxed{\text{echelon form}}$

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$\bullet$ = nonzero

$*$ = anything

$$\begin{bmatrix} 0 & \bullet \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \bullet & * \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} \bullet & * \\ 0 & \bullet \\ 0 & 0 \end{bmatrix}$$

The following is in echelon form:

4 x 10 matrix

$$\begin{bmatrix} 0 & 0 & \boxed{5} & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \boxed{6} & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{7} & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \boxed{9} \end{bmatrix}$$

$\boxed{\cdot}$ = leading entry

Here each $*$ can be replaced by an arbitrary number.

The matrix $\begin{bmatrix} 0 & 0 & \boxed{5} & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \boxed{6} & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is also in echelon form.

Examples **NOT** in echelon form

The matrix $\begin{bmatrix} 0 & 0 & 0 & 6 & * & * & * & * & * & * \\ 0 & 0 & 5 & 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$ is **not** in echelon form.

The matrix $\begin{bmatrix} 0 & 0 & 6 & * & * & * & * & * & * & * \\ 0 & 0 & 5 & * & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 9 \end{bmatrix}$ is **not** in echelon form.

A sort of degenerate case: every one-row matrix is in echelon form. (Why?)

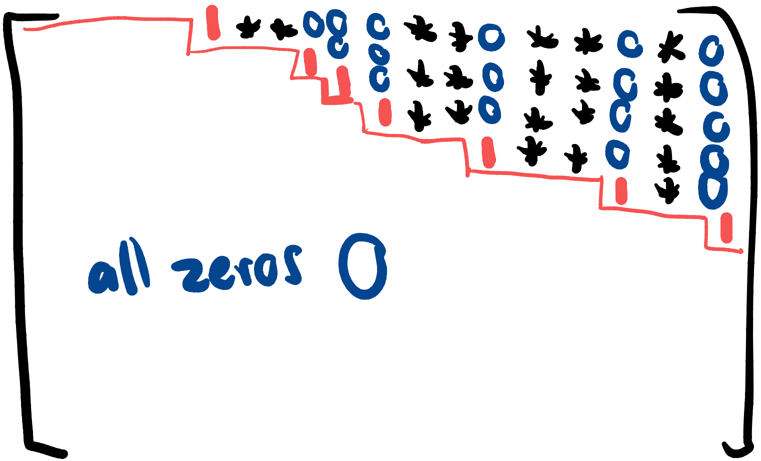
The only one-column matrices in echelon form are ones like $\begin{bmatrix} * \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

Next: reduced echelon form

picture of [a matrix in reduced echelon form

● = nonzero entry

* = any entry



generic picture

Concrete example:

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There is a more restrictive version of echelon form that will be useful.

Definition. A matrix in echelon form is *reduced* if

(R1) Each nonzero row has leading entry 1.

(R2) The leading 1 in each nonzero row is the only nonzero number in its column.

A matrix in echelon form that is reduced is said to be in *reduced echelon form*.

The following matrix is in echelon form but is not reduced:

rescale

$$\begin{bmatrix} 0 & 0 & 1 & 0 & * & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

replacements

But we can apply row operations to turn it into reduced echelon form:

$$\begin{bmatrix} 0 & 0 & 1 & 0 & * & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 1 & * & * & 0 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The fundamental theorem of today is the following:

Theorem. Each matrix A is row equivalent to exactly one matrix $\mathsf{RREF}(A)$ in reduced echelon form.

The proof of this result is included in an appendix of the textbook.

We call $\boxed{\mathsf{RREF}(A)}$ the *reduced echelon form* of A.

Before describing how to compute $\mathsf{RREF}(A)$, we focus on what $\mathsf{RREF}(A)$ tells us about the linear system that has A as its augmented matrix.

Ex. If $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \xrightarrow[\text{to row 2}]{\text{add } -2 \times \text{row 1}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow[\text{to row 1}]{\text{add } -1 \times \text{row 2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathsf{RREF}(A)$

2.2 Solving a linear system from reduce echelon form

A *pivot position* in a matrix A is the location containing a leading 1 in $\mathsf{RREF}(A)$.

We sometimes refer to an entry in a pivot position of a matrix as a *pivot*.

A *pivot column* in a matrix A is a column containing a pivot position.

For a linear system in variables $x_1, x_2, \ldots, x_n$ with augmented matrix matrix A, the variable x_i is *basic* if i is a pivot column of A and is *free* otherwise.

Example. If a linear system has augmented matrix A with

$$\mathsf{RREF}(A) = \begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{then the system is equivalent to} \quad \begin{cases} x_1 - 5x_3 = 1 \\ x_2 + x_3 = 4 \\ 0 = 0. \end{cases}$$

The pivot columns of A are 1 and 2, so the basic variables are x_1 and x_2. The only free variable is x_3.

free

basic

For system $\begin{cases} x_1 - 5x_3 = 1 \\ x_2 + x_3 = 4 \\ 0 = 0 \end{cases}$ the augmented matrix is in RREF form already.

Such systems are easy to solve: each equation expresses a basic variable in terms of free vars.

so just pick any values for free then solve for the basic vars using these eqs.

general solution is $\underbrace{(x_1, x_2, x_3)}_{\text{basic}}$ $\overset{\text{free}}{}$ $= (1+5a, 4-a, a)$ for any $a \in \mathbb{R}$

using eqs in linear system whose aug. matrix is RREF(A)

To find all solutions to the system, **choose any values for the free variables** and then **solve for the basic variables**.

In the above system, we have $x_1 = 5x_3 + 1$ and $x_2 = 4 - x_3$.

Hence all solutions have the form $(s_1, s_2, s_3) = (5a + 1, 4 - a, a)$ for $a \in \mathbb{R}$.

Theorem. Consider a linear system whose augmented matrix is $\boxed{A}$.

- The system has 0 solutions if the last column of A contains a pivot.

  In this case RREF(A) has a row of the form $\begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}$ so our system is equivalent to a linear system containing the false equation $0 = 1$.

- The system has only 1 solution if there are $\underline{\text{no free variables}}$ and the last column is not a pivot.

- Otherwise, the system has infinitely many solutions. (at least one free var)

in this case RREF(A) will look like

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & x \\ 0 & 1 & 0 & 0 & y \\ 0 & 0 & 1 & 0 & z \\ 0 & 0 & 0 & 1 & a \end{array}\right]$$

last col gives the unique solution

Once we have computed $\mathsf{RREF}(A)$ and identified the free and basic variables, we can write down all solutions to the system (if there are solutions) exactly as in the example above: by letting each free variable be arbitrary, and then solving for the basic variables in terms of the free variables.

2.3 Computing reduced echelon form using row operations

The last part of today's lecture covers the least interesting aspect of the reduced echelon form of a matrix: how to actually compute it. We first present the relevant algorithm for one chosen example.

algorithm has two parts:

1. (harder) convert A to (some) echelon form
2. (easier) convert echelon form to reduced form

$$\begin{cases} 3x_2 - 6x_3 = 6 \\ 3x_1 - 7x_2 + 8x_3 = -5 \\ 3x_1 - 9x_2 + 12x_3 = -9 \end{cases}$$

augmented matrix

Example (Row reduction to echelon form, for a specific matrix)**.**

Input: for the general algorithm, the input is an $m \times n$ matrix A.

Suppose this matrix is

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 \\ 3 & -7 & 8 & -5 \\ 3 & -9 & 12 & -9 \end{bmatrix}$$

Procedure:

1. Begin with the leftmost nonzero column.

   This is a pivot column. The pivot position is the top position of the column.

   For our matrix, the leftmost nonzero column is the first column; the pivot nonzero column

position is boxed:

$$\begin{bmatrix} \boxed{\mathbf{0}} & 3 & -6 & 6 \\ \mathbf{3} & -7 & 8 & -5 \\ \mathbf{3} & -9 & 12 & -9 \end{bmatrix}.$$

2. Select a nonzero entry in the current pivot column.

   If needed, perform a row operation to swap the row with this entry and the top row.

   For example, we can select the 3 in the second row of the first column and then swap rows 1 and 2:

$$\begin{bmatrix} \boxed{\mathbf{0}} & 3 & -6 & 6 \\ \mathbf{3} & -7 & 8 & -5 \\ \mathbf{3} & -9 & 12 & -9 \end{bmatrix} \to \begin{bmatrix} \boxed{3} & -7 & 8 & -5 \\ 0 & 3 & -6 & 6 \\ 3 & -9 & 12 & -9 \end{bmatrix}.$$

replacement

3. Use row operations to create zeros below the boxed pivot position:

$$\begin{bmatrix} \boxed{3} & -7 & 8 & -5 \\ 0 & 3 & -6 & 6 \\ 3 & -9 & 12 & -9 \end{bmatrix} \to \begin{bmatrix} \boxed{3} & -7 & 8 & -5 \\ 0 & 3 & -6 & 6 \\ 0 & -2 & 4 & -4 \end{bmatrix}$$

add $-1 \times$ row 1 to row 3

4. Repeat steps 1-3 on the bottom right submatrix:

$$\left[\begin{array}{c|ccc} 3 & -7 & 8 & -5 \\ \hline 0 & \boxed{3} & -6 & 6 \\ 0 & \cancel{-2} & 4 & -4 \end{array}\right] \to \left[\begin{array}{c|ccc} 3 & -7 & 8 & -5 \\ \hline 0 & \boxed{3} & -6 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{c|c|cc} 3 & -7 & 8 & -5 \\ \hline 0 & 3 & -6 & 6 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]$$

5. We now have a matrix in echelon form: $\begin{bmatrix} \boxed{3} & -7 & 8 & -5 \\ 0 & \boxed{3} & -6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Start with the row containing the rightmost pivot position in our matrix, now in echelon form.

ⓐ rescale nonzero rows to have leading entries all 1

ⓑ do replacements to create zeros above each leading 1

Rescale rightmost pivot, then cancel entries above rightmost pivot position in same column:

$$\begin{bmatrix} 3 & -7 & 8 & -5 \\ 0 & \boxed{3} & -6 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{ⓐ} \begin{bmatrix} 3 & \cancel{-7} & 8 & -5 \\ 0 & \boxed{1} & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{ⓑ} \begin{bmatrix} 3 & 0 & -6 & 9 \\ 0 & \boxed{1} & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Repeat with the next pivot position, going right to left:

$$\begin{bmatrix} \boxed{3} & 0 & -6 & 9 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{ⓐ} \begin{bmatrix} \boxed{1} & 0 & -2 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The result is the reduced echelon form

$$\boxed{\mathsf{RREF}(A) = \begin{bmatrix} \boxed{1} & 0 & -2 & 3 \\ 0 & \boxed{1} & -2 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}.$$

Thus:
the linear system corresponding to starting matrix has infinitely many solutions $\leftarrow$

- pivot columns: 1 and 2 (NOT 4)
- basic vars: x_1, x_2
- free vars: x_3

The steps to compute $\mathsf{RREF}(A)$ for a generic matrix A are similar to above.

Algorithm (Row reduction to echelon form, for a generic matrix)**.**

Input: an $m \times n$ matrix A.

Procedure:

1. Begin with the leftmost nonzero column.

   This is a pivot column. The pivot position is the top position of the column.

2. Select a nonzero entry in the current pivot column. If needed, perform a row operation to swap the row with this entry and the top row.

3. Use row operations to create zeros in the entries below the pivot position.

4. Cover the row containing the current pivot position, and then apply the previous steps to the $(m-1) \times n$ submatrix that remains. Repeat until the entire matrix is in echelon form.

convert echelon form to reduced:

5. Start with the row containing the rightmost pivot position in our matrix, now in echelon form.

   Use row operations to rescale this row to have leading entry 1.

   Then use row operations to create zeros in the entries in the same column above each leading entry.

   Repeat this for each successive pivot position going left, until the matrix is in reduced echelon form.

Output: $\mathsf{RREF}(A)$.

Observation. If a matrix E is in echelon form and is row equivalent to A, then we say that E is *an echelon form* of A. In any echelon form E of a matrix A, the locations of the leading entries are the same. This means we can compute the pivot positions of A from any echelon form E, and to find the pivots we only have to go through step 4 in the previous algorithm.

to find pivots in A