# MATH 2121 — Lecture #4

Outline for today:

1. Review: vectors, linear combinations, span
2. Matrix-vector multiplication, matrix eqs
3. Linear independence

Announcements: online HW2 + offline HW2

## 1 Last time: Vectors

matrix with 1 column

A *(column)* $\boxed{\textit{vector}}$ of size $n$ is an $n \times 1$ matrix: $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$.

Let $\boxed{\mathbb{R}^n}$ be the set of all vectors with exactly $n$ rows.

$+$ We can add two vectors of the same size: $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$.

two vector operations: $+$ and $\cdot$

$$\begin{bmatrix}1\\2\\0\end{bmatrix}+\begin{bmatrix}3\\5\\7\end{bmatrix}=\begin{bmatrix}4\\7\\7\end{bmatrix}$$

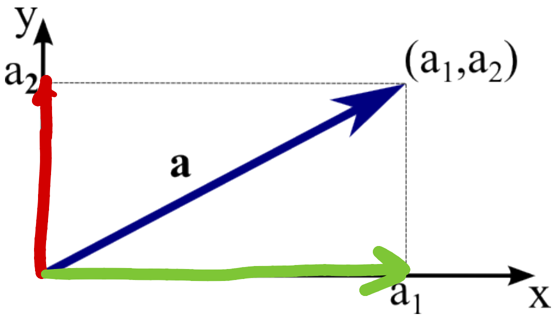
means "number"

$$6\begin{bmatrix}1\\3\\0\\2\end{bmatrix}=\begin{bmatrix}6\\18\\0\\12\end{bmatrix}$$

- We can multiply a vector by a $\boxed{\textit{scalar}}$: $cv = c\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix} = \begin{bmatrix}cv_1\\cv_2\\\vdots\\cv_n\end{bmatrix}$ for $c \in \mathbb{R}$.

We draw $a = \begin{bmatrix}a_1\\a_2\end{bmatrix} \in \mathbb{R}^2$ as arrow in Cartesian plane from origin to $(x, y) = (a_1, a_2)$:

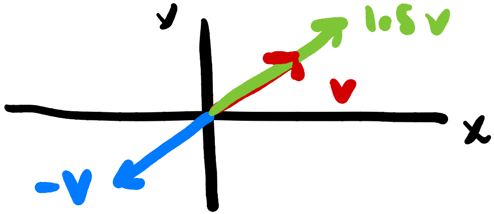
$$\uparrow = \begin{bmatrix}0\\a_2\end{bmatrix}$$



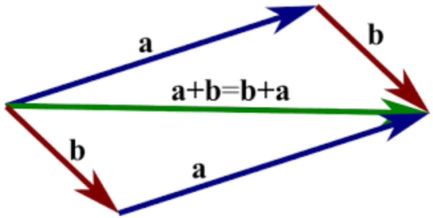

$$\rightarrow = \begin{bmatrix}a_1\\0\end{bmatrix}$$

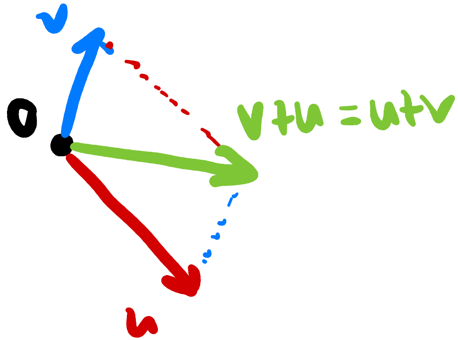
Can also draw vectors in $\mathbb{R}^3$ as arrows

## Scalar multiplication:

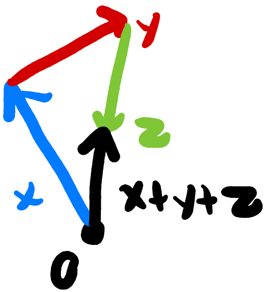


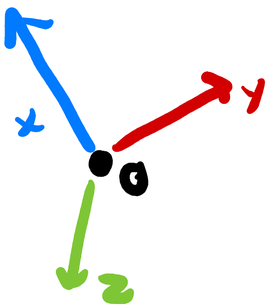

Relative to this picture, the sum $a+b$ of two vectors $a, b \in \mathbb{R}^2$ is the arrow from the origin to the opposite vertex of the parallelogram with sides $a$ and $b$:




The *zero vector* $\boxed{0 \in \mathbb{R}^n}$ is the vector $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. We always have $0 + v = v + 0 = v$.




what is $x + y + z$ ?

linear system $\longleftrightarrow$ augmented matrix $\longleftrightarrow$ vector equation

$$\begin{cases} 2x_1 + 3x_2 = 7 \\ 4x_2 + 5x_2 = 0 \\ x_1 + 2x_2 = 1 \end{cases}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 7 \\ 4 & 5 & 0 \\ 1 & 2 & 1 \end{array}\right]$$

$$[\, v_1 \; v_2 \mid b \,]$$

$$x_1 \underbrace{\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}}_{v_1} + x_2 \underbrace{\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}}_{v_2} = \underbrace{\begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}}_{b}$$

have same solutions

given vectors (all of same size)

A *linear combination* of vectors $\underbrace{v_1, v_2, \ldots, v_p}$ $\in \mathbb{R}^n$ is any vector of the form

$y = c_1v_1 + c_2v_2 + \cdots + c_pv_p \in \mathbb{R}^n$ for any choice of numbers $c_1, c_2, \ldots, c_p \in \mathbb{R}$. given scalars

The *span* of vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ is the set of all of their linear combinations.

We denote this set by (usually an infinite set)

$$\mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\} \qquad \text{or} \qquad \text{span}\{v_1, v_2, \ldots, v_p\}.$$

**Proposition.** If $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$, then a vector $y \in \mathbb{R}^n$ belongs to $\mathbb{R}$-span$\{v_1, v_2, \ldots, v_p\}$ if and only if the $n \times (p+1)$ matrix $\underbrace{\begin{bmatrix} v_1 & v_2 & \ldots & v_p & y \end{bmatrix}}_{A}$ is the augmented matrix of a consistent linear system.

$\Updownarrow$

RREF(A) has no pivot position in last column

Ex.

$$\begin{array}{ccccccccc} & v_1 & & v_2 & & v_3 & & y \\ 2 & \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} & +3 & \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix} & +(-2) & \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} & = & \begin{bmatrix} 9 \\ 17 \\ 6 \end{bmatrix} \end{array}$$

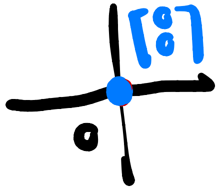
$y$ is a linear combination of $v_1, v_2, v_3$

for vectors in $\mathbb{R}^2$, the span is either a point, a line, or (the whole) plane.

In terms of geometry, the span of a set of vectors in $\mathbb{R}^2$ is either a point (at the origin), a line (through the origin), or the whole plane $\mathbb{R}^2$.

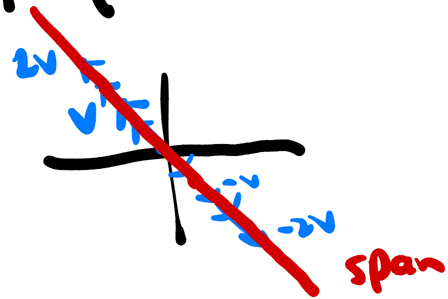
point: $\mathbb{R}\text{-span}\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\} = \left\{c\begin{bmatrix}0\\0\end{bmatrix} : c \in \mathbb{R}\right\} = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$

(only one way) picture:




line: $\mathbb{R}\text{-span}\left\{\begin{bmatrix}v_1\\v_2\end{bmatrix} = v\right\} = \{cv \mid c \in \mathbb{R}\}$ if $v \neq \begin{bmatrix}0\\0\end{bmatrix}$

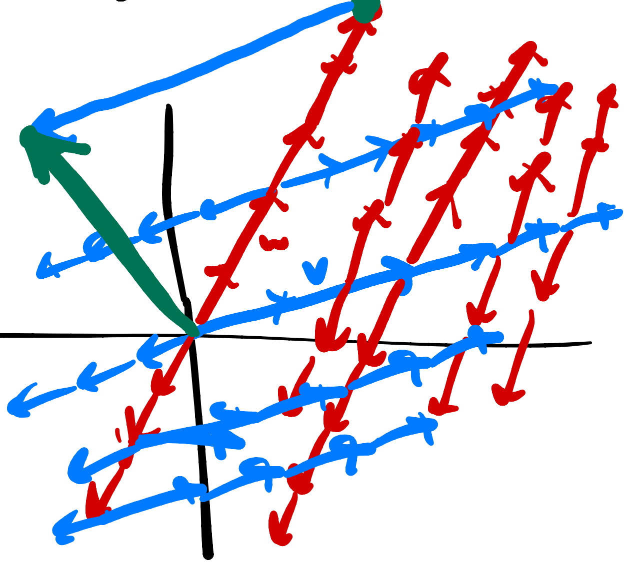
(many ways) picture:



The span of a set of vectors in $\mathbb{R}^3$ is either a point (at the origin), a line (through the origin), a plane (containing the origin), or all of $\mathbb{R}^3$.

plane : $\mathbb{R}$-span$\left\{ \begin{array}{c} \text{any two vectors not on} \\ \text{same line: } v \text{ and } w \end{array} \right\} = \mathbb{R}^2$

(many ways)




$\left. \begin{array}{l} v = 1v + 0w \\ w = 0v + 1w \end{array} \right\}$ in span

## 2 Multiplying matrices and vectors

We have been using matrices as a compact notation for representing linear systems.

Today we introduce a second way of viewing a matrix: namely, as an $\boxed{\text{operator}}$ that transforms one vector to another.

Ex How to multiply an $m \times n$ matrix $A$ with $v \in \mathbb{R}^n$

$$\overset{A}{\begin{bmatrix} 1 & 3 & 7 & 0 \\ 1 & 5 & 8 & 1 \\ 2 & 4 & 0 & 0 \end{bmatrix}} \overset{v}{\begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix}} \overset{\text{def}}{=} 2\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} + 4\begin{bmatrix} 7 \\ 8 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 33 \\ 42 \\ 8 \end{bmatrix}$$

$m = 3$, $n = 4$

result is in $\mathbb{R}^m = \mathbb{R}^3$

[$Av$ is a linear comb of columns of $A$]

$A$ is $m \times n$ matrix, $A = [\vec{a}_1 \; \vec{a}_2 \; \cdots \; \vec{a}_n]$

$v \in \mathbb{R}^n$, $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$, $Av \in \mathbb{R}^m$

**Definition.** If $A$ is a matrix with columns $a_1, a_2, \ldots, a_n \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$, so that

$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \end{bmatrix} \qquad \text{and} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

then the *matrix-vector product* $Av$ is the vector in $\mathbb{R}^m$ given by:

$$\boxed{Av = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1\vec{a}_1 + v_2\vec{a}_2 + \cdots + v_n\vec{a}_n \in \mathbb{R}^m.}$$

Thus $Av$ is the linear combination of the columns of $A$ with coefficients given by the entries of $v$.

multiplying 2x3 matrix by 3x1 vector to get a 2x1 vector

**Example.** If $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix}$ and $v = \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$ then $a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, and $a_3 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$ so

$$\boxed{Av = 4a_1 + 3a_2 + 7a_3 = \underbrace{\begin{bmatrix} 4 \\ 0 \end{bmatrix}}_{4a_1} + \underbrace{\begin{bmatrix} 6 \\ -15 \end{bmatrix}}_{3a_2} + \underbrace{\begin{bmatrix} -7 \\ 21 \end{bmatrix}}_{7a_3} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}} = Av \in \mathbb{R}^2$$

**Example.** If $A = \begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix}$ and $v = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$ then $a_1 = \begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix}$ and $a_2 = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$ so we have

$$\boxed{Av = 4a_1 + 7a_2 = 4\begin{bmatrix} 2 \\ 8 \\ -5 \end{bmatrix} + 7\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 8 \\ 32 \\ -20 \end{bmatrix}}_{4a_1} + \underbrace{\begin{bmatrix} -21 \\ 0 \\ 14 \end{bmatrix}}_{7a_2} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}} = Av \in \mathbb{R}^3$$

multiplying 3x2 matrix with a 2x1 vector to get a 3x1 vector

**Comments** $Av$ is only defined when

(# columns of $A$) = (# rows of $v$)

in case, (# rows of $Av$) = (# rows of $A$)

If $A$ is $m \times n$ then $Av$ is only defined for $v \in \mathbb{R}^n$, and in this case $Av \in \mathbb{R}^m$.

Thus $\boxed{A \text{ transforms vectors in } \mathbb{R}^n \text{ to vectors in } \mathbb{R}^m.}$

This transformation is *linear*:

1. If $A$ is an $m \times n$ matrix and $u, v \in \mathbb{R}^n$ then $\boxed{A(u+v) = Au + Av.}$
2. If $A$ is an $m \times n$ matrix and $v \in \mathbb{R}^n$ and $c \in \mathbb{R}$ then $\boxed{A(cv) = c(Av)}$

if $A = [\vec{a}_1 \ \vec{a}_2]$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

then $A(v+w) = A\begin{bmatrix} v_1+w_1 \\ v_2+w_2 \end{bmatrix} = (v_1+w_1)\vec{a}_1 + (v_2+w_2)\vec{a}_2$

$$= v_1\vec{a}_1 + w_1\vec{a}_1 + v_2\vec{a}_2 + w_2\vec{a}_2$$

$$= (\underbrace{v_1\vec{a}_1 + v_2\vec{a}_2}_{=Av}) + (\underbrace{w_1\vec{a}_1 + w_2\vec{a}_2}_{=Aw}) = Av + Aw$$

Why "linear": multiplication by $A$ transforms lines to lines

a line is the span of one nonzero vector $L = \mathbb{R}\text{-span}\{v\}$

then define $AL \overset{\text{def}}{=} \{Aw \mid w \in L\} = \mathbb{R}\text{-span}\{Av\}$ another line

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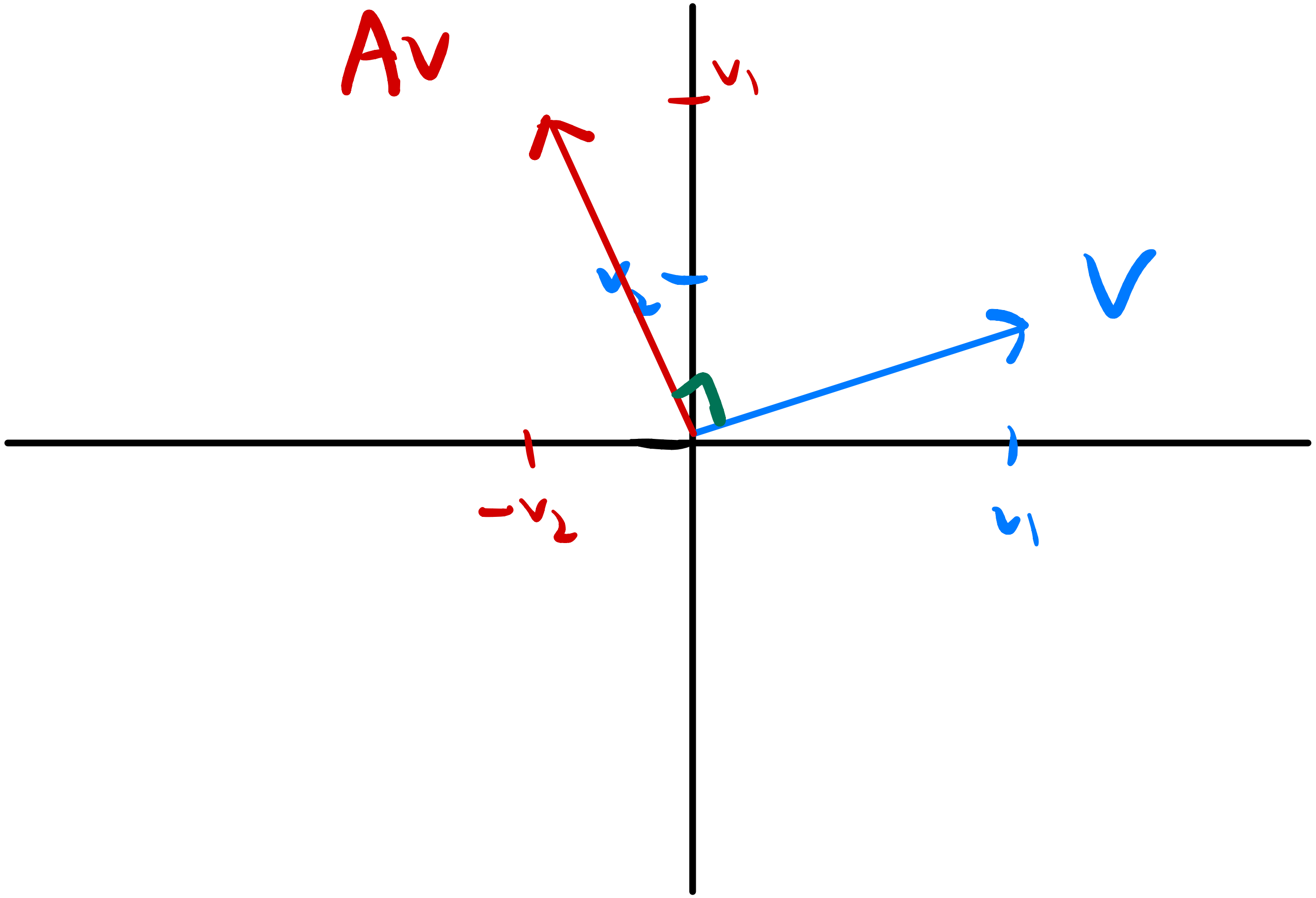
Often we can interpret multiplication by 2x2 matrices geometrically

Ex If $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ then

$$A\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + v_2\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ v_1 \end{bmatrix} + \begin{bmatrix} -v_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -v_2 \\ v_1 \end{bmatrix}$$

The vector $Av$ is just $v$ rotated $90^\circ$ CCW




$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Let $A$ and $v$ be the general $m \times n$ matrix and $n$ row vector given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

To compute $Av$: match up entries in $i$th column of $A$ with the $i$th row of $v$.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 + \cdots + a_{1n}v_n \\ a_{21}v_1 + a_{22}v_2 + \cdots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + a_{m2}v_2 + \cdots + a_{mn}v_n \end{bmatrix}$$

For example, $\overset{A}{\underset{1\times 4}{\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}}} \underset{\substack{v \\ 4\times 1}}{\begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}} = 1\cdot 5 + 2\cdot 6 + 3\cdot 7 + 4\cdot 8 = 5 + 12 + 21 + 32 = \boxed{70}.$

$$\underbrace{\begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix}}_{A} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \qquad \begin{bmatrix} \\ Av \\ \\ \end{bmatrix}$$

# columns of A = # rows of v

# rows of Av = # rows of A

$A = [a_1 \ a_2 \ \cdots \ a_n]$

$\boxed{Ax = b}$ a matrix equation

## 3 Matrix equations

If $A$ is an $m \times n$ matrix with columns $a_1, a_2, \ldots, a_n \in \mathbb{R}^m$ and

vector of variables

$$\boxed{x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} \quad \text{and} \quad \boxed{b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \in \mathbb{R}^m}$$

vector of numbers

where each $x_i$ is a variable, then we call $Ax = b$ a *matrix equation*.

**Proposition.** The matrix equation $Ax = b$ has the same solutions as both the vector equation $x_1a_1 + x_2a_2 + \cdots + x_na_n = b$ and the linear system whose augmented matrix is $\begin{bmatrix} a_1 & a_2 & \ldots & a_n & b \end{bmatrix} = [A \mid b]$

* **Proposition.** The matrix equation $Ax = b$ has a solution if and only if $b$ is a linear combination of the columns of $A$, that is, $b \in \mathbb{R}\text{-span}\{a_1, a_2, \ldots, a_n\}$.

linear system

$$\begin{cases} x_1 + 3x_2 - x_3 = 0 \\ x_1 + x_2 + x_3 = 5 \end{cases}$$

augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & 5 \end{array}\right]$$

matrix equation

$$Ax = b$$

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

(coefficient matrix)

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$b = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

vector equation

$$x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

**Example.** Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$.

Does $Ax = b$ have a solution for all choices of $b_1, b_2, b_3 \in \mathbb{R}$?

The system $\boxed{Ax = b}$ has a solution if and only if

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix}$$

aug matrix of a linear system (same solutions as $Ax=b$)

is the augmented matrix of a consistent linear system. We can determine if this system is consistent by row reducing the matrix to echelon form:

in echelon form

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 + b_2 \\ 0 & 7 & 5 & 3b_1 + b_3 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 3 & 4 & b_1 \\ 0 & \boxed{14} & 10 & 4b_1 + b_2 \\ 0 & 0 & 0 & \boxed{b_1 - \frac{1}{2}b_2 + b_3} \end{bmatrix}.$$

maybe leading entry (if nonzero)

we want to know if last column has a pivot as this would mean $Ax=b$ has no solution

The last matrix is in echelon form, so its leading entries are the pivot positions of our first matrix. The linear system is consistent if and only if the last column does not contain a pivot position. This occurs precisely when $\boxed{b_1 - \frac{1}{2}b_2 + b_3 = 0.}$

But we can choose numbers such that $b_1 - \frac{1}{2}b_2 + b_3 \neq 0$: take $b_1 = 1$ and $b_2 = b_3 = 0$. Therefore our original matrix equation $\boxed{Ax = b}$ does <u>not always</u> <u>have a</u> <u>solution</u>.

only if this holds does

$Ax = b$

have a solution

We can generalize this example:

**Theorem.** Let $A$ be an $m \times n$ matrix. The following properties are equivalent, meaning that if one of them holds, then they all hold, but if one of them fails to hold, then they all fail:

1. For each vector $b \in \mathbb{R}^m$, the matrix equation $Ax = b$ has a solution.
2. Each vector $b \in \mathbb{R}^m$ is a linear combination of the columns of $A$.
3. The span of the columns of $A$ is the set $\mathbb{R}^m$.

   (Say this as: "the columns of $A$ span $\mathbb{R}^m$".)
4. $A$ has a pivot position in every row. ← not obviously equivalent, but computable

Pf idea: 1 2 3 correspond to when

$[A \mid b]$ has no pivot in last column for all $b$.

This is only guaranteed if $\mathrm{RREF}(A) = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \cdots \end{bmatrix}$ has pivots in all rows as if RREF(A) has pivots in all rows then

$$\mathrm{RREF}([A \mid b]) = \left[\ \mathrm{RREF}(A) \;\middle|\; \begin{matrix} ? \\ \vdots \\ ? \end{matrix}\ \right]$$

$$= \left[\begin{matrix} 1 & \cdots & - & - & - \\ & & 1 & \cdots & - \\ & & & 1 & \cdots \end{matrix} \;\middle|\; \begin{matrix} ? \\ \vdots \\ ? \end{matrix}\right]$$

no pivots here

## 4 Linear independence

Let $v_1, v_2, \ldots, v_p$ be vectors in $\mathbb{R}^n$.

These vectors are *linearly independent* if the **only solution** to the vector equation

$$x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$$

is given by $\boxed{x_1 = x_2 = \cdots = x_p = 0.}$

The vectors $v_1, v_2, \ldots, v_p$ are *linearly dependent* otherwise, that is, if there are numbers $c_1, c_2, \ldots, c_p \in \mathbb{R}$, **at least one of which is nonzero**, such that

$$c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0.$$

**Example.** If $v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Then $v_1 + v_3 = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$ and $v_2 + v_3 = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$, so

$$2(v_1 + v_3) - (v_2 + v_3) = 2v_1 - v_2 + v_3 = 0.$$

Hence $v_1, v_2, v_3$ are linearly dependent.

It is usually not so easy to determine whether a given list of vectors is linearly independent or not. The following result gives a general way to check this:

**Theorem.** The columns of a matrix $A$ are linearly independent if and only if $A$ has a pivot position in every column.

**Corollary.** Let $v_1, \dots, v_p \in \mathbb{R}^n$. If $p > n$ then the vectors are linearly dependent.

*Proof.* The $n \times p$ matrix $A = \begin{bmatrix} v_1 & v_2 & \dots & v_p \end{bmatrix}$ has at most $\min(n, p)$ pivot columns, because each column contains at most one pivot position, and each row contains at most one pivot position. Therefore if $p > n$ then $A$ does not have a pivot position in every column so its columns are linearly dependent. $\square$