MATH 2121 — Lecture #5

Outline:

1. Review – matrix equations
2. Linear independence in more depth
3. Key idea: matrices $\longleftrightarrow$ linear transformations

Announcements: another round of HW assignments you can multiply $m \times n$ matrix A with $n \times 1$ vector v to get $m \times 1$ vector "Av"

1 Last time: multiplying vectors and matrices

Given a matrix $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ and vector $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ define

$$\boxed{Av = v_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + v_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + v_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \in \mathbb{R}^m.}$$

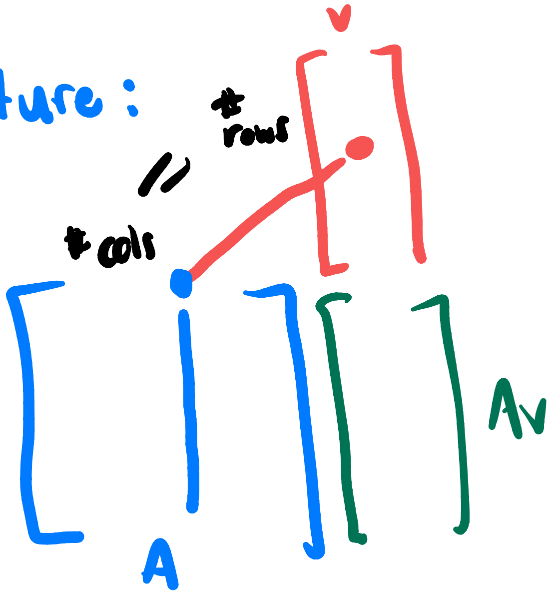
the span of the columns of A is $\{Ax \mid x \in \mathbb{R}^n\}$

Call Av the product of A and v, or the vector given by multiplying v by A.

Example. $\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{v} = -1 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 6 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} -1+0+3 \\ -5+0+7 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 2 \end{bmatrix}}_{Av}.$

Rmk Av is a linear combination of the columns of A and any such lin. comb. is given by (A times some vector)

Picture:



#rows of Av = #rows of A

vector of n variables

③ also means RREF(A) has no zero rows

If A is an $m \times n$ matrix, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $b \in \mathbb{R}^m$, then $\boxed{Ax = b}$ is a *matrix equation*.

$\boxed{Ax = b \text{ has the same solutions as linear system with augmented matrix } \left[\begin{array}{cc} A & b \end{array}\right].}$

Theorem. Let A be an $m \times n$ matrix. The following are equivalent:

1. $Ax = b$ has a solution for any $b \in \mathbb{R}^m$.
2. The span of the columns of A is all of $\mathbb{R}^m$.
3. A has a pivot position in every row.

coeff matrix $= A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 4 & 5 & 1 \end{bmatrix}$

aug. matrix $= [A|b]$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 4 & 5 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_n \end{bmatrix} = \overbrace{\begin{bmatrix} 7 \\ 2 \end{bmatrix}}^{b} \quad \xleftrightarrow[\text{solutions}]{\text{same}} \quad \begin{cases} x_1 + 2x_2 + 3x_3 + 2x_4 = 7 \\ 2x_1 + 4x_2 + 5x_3 + x_4 = 2 \end{cases}$$

linear system in 2 eqs, 4 vars

why does "pivot in every row" $\Longleftrightarrow$ "$Ax=b$ has a solution for any b"

① If A has pivot in every row then $\mathrm{RREF}(A) = \begin{bmatrix} 0&0&0&1&?&?&?&?&? \\ 0&0&0&0&0&1&?&?&? \\ 0&0&0&0&0&0&0&0&1 \end{bmatrix}$

and this means $\mathrm{RREF}([A\ b]) = \left[\begin{array}{c|c} \underbrace{\begin{matrix} 0&0&0&1&?&?&?&?&? \\ 0&0&0&0&0&1&?&?&? \\ 0&0&0&0&0&0&0&0&1 \end{matrix}}_{\mathrm{RREF}(A)} & \underbrace{\begin{matrix} ? \\ ? \\ ? \end{matrix}}_{??} \end{array}\right]$. We

don't know what $\begin{matrix} ? \\ ? \\ ? \end{matrix}$ is, but

there are NO PIVOTS IN LAST COLUMN. and this means $Ax=b$ has a solution

② If A lacks a pivot in some row then $\mathrm{RREF}(A) = \begin{bmatrix} 0&0&0&1&?&?&?&? \\ 0&0&0&0&0&1&?&? \\ 0&0&0&0&0&0&0&0 \end{bmatrix}$

so $\mathrm{RREF}([A\ b]) = \begin{bmatrix} 0&0&0&1&?&?&?&?&z_1 \\ 0&0&0&0&0&1&?&?&z_2 \\ 0&0&0&0&0&0&0&0&z_3 \end{bmatrix}$ where z_1, z_2, z_3 depend on b

but for some b we can make $z_3 \neq 0$ so pivot in last col, so

$Ax=b$ has no solution (for some special b)

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$$

Does $Ax = b$ always have a solution for all $b \in \mathbb{R}^3$?

Example. The matrix equation

$$\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

may fail to have a solution since

$$\mathrm{RREF}(A) = \mathsf{RREF}\left(\begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix} \leftarrow \text{zero row}$$

has pivot positions only in rows 1 and 2.

$\boxed{\text{NO}}$

as there are not pivots in every row of A

Terminology: Choose vectors $v_1\ v_2\ v_3 \cdots v_p$ in $\mathbb{R}^n$

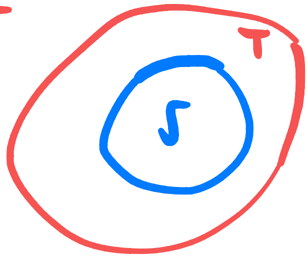
$y = c_1 v_1 + c_2 v_2 + \dots + c_p v_p$ is a <u>linear combination</u> of our vectors if $c_1, c_2, \dots, c_p \in \mathbb{R}$

The span of $v_1\ v_2 \cdots v_p$ is the (infinite) set of all of their linear combinations. Denoted $\mathbb{R}\text{-span}[v_1, v_2, \dots, v_p]$

<u>**Notice:**</u> $$\underbrace{\mathbb{R}\text{-span}[v_1, v_2, \dots, v_i]}_{= S} \subseteq \underbrace{\mathbb{R}\text{-span}[v_1, v_2, \dots, v_i, v_{i+1}]}_{= T}$$

$\subseteq$ means "is contained in"

(the sets could also be $=$)



2 Linear independence

We briefly introduced the notion of linear independence last time.

Suppose we have some vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$.

Notation: For sets S and T write $S \subseteq T$ to mean that every element of S is in T.

Recall that the *span* of a set of vectors is the set of all possible linear combinations of the given vectors. If you have a smaller set of vectors inside a bigger set, then the span of the smaller set is always contained in the span of the bigger set.

This means that if $y \in \mathbb{R}^n$ is any vector then

$$\mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\} \subseteq \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p, y\}.$$

When is this containment equality?

When is it *strict* (meaning the two sides are not equal)?

[strict]

$\boxed{\text{Case 1}}$ If y is **not** a linear combination of $v_1, v_2, \ldots, v_p$ then

$$\mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\} \neq \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p, y\}$$

since y is in the span on the right but **not** on the left.

[equal]

$\boxed{\text{Case 2}}$ Suppose y is a linear combination of $v_1, v_2, \ldots, v_p$. This means that

$$y = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p \quad \text{for some } c_i \in \mathbb{R}.$$

Then every element of $\mathbb{R}$-span$\{v_1, \ldots, v_p, y\}$ is **also** in $\mathbb{R}$-span$\{v_1, \ldots, v_p\}$, since

$$\underbrace{a_1 v_1 + a_2 v_2 + \cdots + a_p v_p + by}_{\in \mathbb{R}\text{-span}\{v_1, v_2 \ldots, v_p, y\}} = \underbrace{(a_1 + bc_1)v_1 + (a_2 + bc_2)v_2 + \cdots + (a_p + bc_p)v_p}_{\in \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\}}.$$

In other words, it holds that $\mathbb{R}$-span$\{v_1, v_2, \ldots, v_p\} \supseteq \mathbb{R}$-span$\{v_1, v_2, \ldots, v_p, y\}$.

But we already know that $\mathbb{R}$-span$\{v_1, v_2, \ldots, v_p\} \subseteq \mathbb{R}$-span$\{v_1, v_2, \ldots, v_p, y\}$.

The only way that $\supseteq$ and $\subseteq$ can both hold is if

$$\mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\} = \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p, y\}.$$

any lin. comb of v_i's & y can be rewritten as a lin. comb. of just the v_i's

New (but equivalent) def of linear independence:

$\subsetneq$ means left side is contained in right side but NOT equal

Write $S \subsetneq T$ to mean that $S \subseteq T$ and $S \neq T$.

Definition. The vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ are *linearly independent* if

$$\{0\} \subsetneq \mathbb{R}\text{-span}\{v_1\} \subsetneq \mathbb{R}\text{-span}\{v_1, v_2\} \subsetneq \mathbb{R}\text{-span}\{v_1, v_2, v_3\} \subsetneq \cdots \subsetneq \mathbb{R}\text{-span}\{v_1, v_2, \ldots, v_p\}.$$

the sequence of spans is strictly increasing

Not the same definition as last lecture, but we shall see in a bit that it is equivalent.

Example. $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent, since

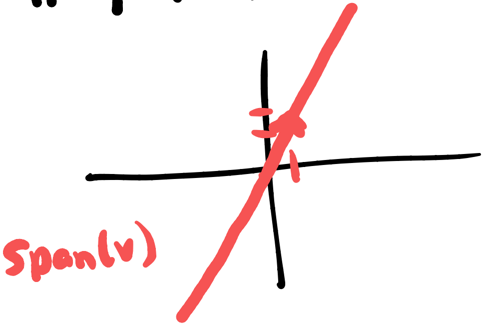
$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \subsetneq \left\{ \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\} \subsetneq \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\} \subsetneq \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} = \mathbb{R}^3$$

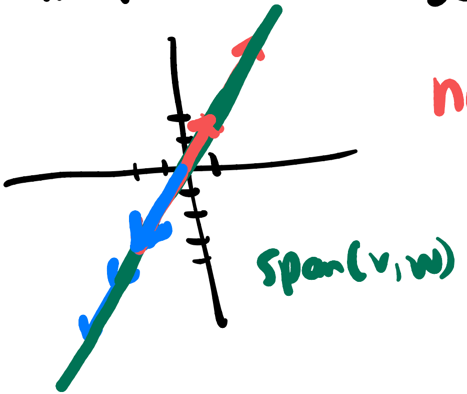
$\text{span}(v_1) \neq \text{span}(v_1, v_2) \neq \text{span}(v_1, v_2, v_3)$

Picture

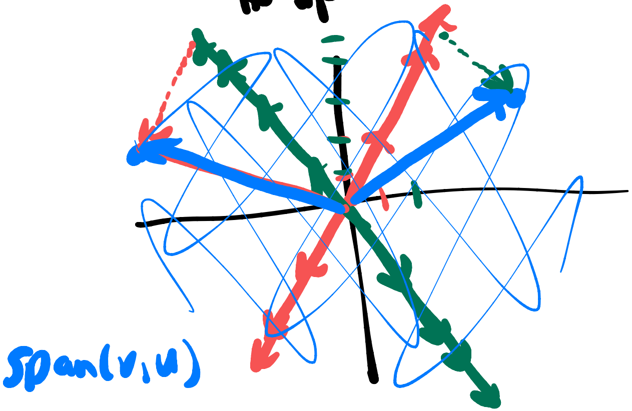
if $v = \begin{bmatrix}1\\2\end{bmatrix}$ and $w = \begin{bmatrix}-2\\-4\end{bmatrix}$ then

$\mathbb{R}^2 \neq \mathbb{R}\text{-span}[v] \;=\; \mathbb{R}\text{-span}[v,w]$ so v, w are not linearly independent





$\mathbb{R}\text{-span}[v,u] = \mathbb{R}^2$



if $u = \begin{bmatrix}2\\-5\end{bmatrix}$

$\Rightarrow$ v, u are linearly independent

Example. $v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are not linearly independent:

$$\mathbb{R}\text{-span}\{v_1, v_2\} = \mathbb{R}\text{-span}\{v_1, v_2, -v_1 - v_2\} = \mathbb{R}\text{-span}\{v_1, v_2, v_3\}.$$

When vectors are not linearly independent, we say they are $\boxed{\textit{linearly dependent}.}$

true that $\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right\} \neq \text{span}(v_1) \neq \text{span}(v_1, v_2)$

but $\text{span}(v_1, v_2) = \text{span}(v_1, v_2, v_3)$

because $v_3 = -v_1 - v_2 \in \text{span}(v_1, v_2)$

Ex $\begin{bmatrix}1\\-1\\0\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0\\2\\-2\end{bmatrix} + \begin{bmatrix}-1\\0\\1\end{bmatrix}$ is a linear dependence among $\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\-2\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$

A *linear dependence* among $v_1, v_2, \ldots, v_p$ is a way of writing the zero vector as

$$0 = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

for some $c_1, c_2, \ldots, c_p \in \mathbb{R}$ that are **not all zero**.

If $0 = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$ is a linear dependence then the matrix equation

a nonzero solution to

$$\begin{bmatrix} v_1 & v_2 & \ldots & v_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = 0$$

has two different solutions given by $(0, 0, \ldots, 0)$ and $(c_1, c_2, \ldots, c_p)$.

(This means the matrix equation has infinitely many solutions — why?)

Proposition (defn of independence from last time)**.** The vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ are *linearly independent* if and only if no linear dependence exists among them.

Algorithm

How to determine if $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ are linearly independent.

- Form the $n \times p$ matrix $A = \begin{bmatrix} v_1 & v_2 & \ldots & v_p \end{bmatrix}$.
- Reduce A to echelon form to find its pivot columns.
- If every column of A has a pivot, then the vectors are linearly independent.
- If some column of A is not a pivot, then the vectors are linearly dependent.

Ex if $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ -4 \end{bmatrix} \longrightarrow A = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$

$\longrightarrow \text{RREF}(A) = \begin{bmatrix} \boxed{1} & -2 \\ 0 & 0 \end{bmatrix}$ only has one pivot $\longrightarrow$ DEPENDENT

Ex if $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ -5 \end{bmatrix} \longrightarrow A \begin{bmatrix} 1 & 2 \\ 2 & -5 \end{bmatrix}$

$\longrightarrow \text{RREF}(A) = \begin{bmatrix} \boxed{1} & 0 \\ 0 & 0 \end{bmatrix}$ has two pivots $\longrightarrow$ INDEPENDENT

Example. The vectors $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix}$ are linearly dependent since

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ -1 & 5 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ 0 & 7 & 21 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(A)$$

where $\sim$ denotes row equivalence.

The last matrix has no pivot position in column 3. In fact, we have

here is a linear dependence :

$$-\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0.$$

so the vectors are linearly dependent

where do $(c_1, c_2, c_3) = (-1, 3, -1)$ come from?

↳ a solution $Ax = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longleftrightarrow \begin{bmatrix} A & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{bmatrix}$

Alternatively: $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$, and $\begin{bmatrix} 5 \\ 9 \\ 15 \end{bmatrix}$ are linearly independent, since

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ -1 & 5 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 3 & 9 \\ 0 & 7 & 20 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathsf{RREF}(A).$$

Every column of A contains a pivot, so three vectors are linearly independent.

consider $\mathrm{RREF}([v])$ which is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

Facts about linear independence.

1. A single vector v is linearly independent if and only if $v \neq 0$.
2. A list of vectors in $\mathbb{R}^n$ is linearly dependent if it includes the zero vector.
3. Vectors $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$ are linearly dependent if and only if some vector v_i is a linear combination of the other vectors $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_p$.

if $v_i = 0$ then column i of $\mathrm{RREF}([v_1, \cdots v_p])$ is also zero

We saw this in the previous example: $\begin{bmatrix} 5 \\ 9 \\ 16 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

4. If $p > n$ then any list of p vectors in $\mathbb{R}^n$ is linearly dependent.

Example. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 60 \end{bmatrix} \in \mathbb{R}^2$ are linearly dependent since $3 > 2$.

3 Linear transformations

A *function* f takes an input x from some set X and produces an output $f(x)$.

We write $\boxed{f : X \to Y}$ to mean that f is a function that takes inputs from the set X and gives outputs that are contained in the set Y.

The set X is called the *domain* of the function f.

The set Y is called the *codomain* of f.

Every element $x \in X$ is a valid input to f.

→ Not every $y \in Y$ needs to occur as an output of f.

Ex the formula $f(x) = x^2$ defines a function $f : \mathbb{R} \to \mathbb{R}$

but no negative number occurs as an output

we are interested in functions with vector inputs in $\mathbb{R}^n$ and vector outputs in $\mathbb{R}^m$: $f: \mathbb{R}^n \to \mathbb{R}^m$

Definition. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function whose domain and codomain are sets of vectors. The function f is a *linear transformation* if both of these properties hold:

(1) $f(u+v) = f(u) + f(v)$ for all vectors $u, v \in \mathbb{R}^n$.

(2) $f(cv) = cf(v)$ for all vectors $v \in \mathbb{R}^n$ and scalars $c \in \mathbb{R}$.

"compatible with vector operations"

Example. If A is an $m \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^m$ is the function with the formula $\boxed{T(v) = Av}$ for $v \in \mathbb{R}^n$ then T is a linear function/transformation

Linear transformations have some additional properties worth noting:

Proposition. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then

(3) $f(0) = 0$.

(4) $f(u - v) = f(u) - f(v)$ for $u, v \in \mathbb{R}^n$.

(5) $f(a_1v_1 + a_2v_2 + \cdots + a_pv_p) = a_1f(v_1) + \cdots + a_pf(v_p)$ for any $a_i \in \mathbb{R}$ and $v_i \in \mathbb{R}^n$.

"compatible with linear combinations"

Geometric intuition: if $S \subseteq \mathbb{R}^n$ is any shape then $f(S) \overset{\text{def}}{=} \{ f(v) \mid v \in S \}$ is another (possible weird) shape inside codomain of f

Idea: if $f: \mathbb{R}^n \to \mathbb{R}^m$ is linear, and $S = (\text{a line in } \mathbb{R}^n)$ then $f(S) = (\text{a line in } \mathbb{R}^m)$

call these elementary (basis) vectors

Define $\underbrace{e_1, e_2, \ldots, e_n} \in \mathbb{R}^n$ as the vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad e_{n-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Fact. If A is an $m \times n$ matrix then Ae_i is the ith column of A.

Proof. Just do the calculation. For example

$$\begin{bmatrix} 1 & 2 & \boxed{\begin{matrix} 3 \\ 7 \end{matrix}} & 4 \\ 5 & 6 & & 8 \end{bmatrix} e_3 = \underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{e_3} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

$$\cancel{0\begin{bmatrix} 1 \\ 5 \end{bmatrix}} + \cancel{0\begin{bmatrix} 2 \\ 6 \end{bmatrix}} + 1\begin{bmatrix} 3 \\ 7 \end{bmatrix} + \cancel{0\begin{bmatrix} 4 \\ 8 \end{bmatrix}}$$

$\square$

Fundamental thm about linear transformations

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation.

1 Then there is a unique $m \times n$ matrix A such that $T(v) = Av$ for all $v \in \mathbb{R}^n$.

2 The matrix A has the formula $A = \left[\ T(e_1) \quad T(e_2) \quad \dots \quad T(e_n)\ \right]$.

Proof. Define $A = \left[\ T(e_1) \quad T(e_2) \quad \dots \quad T(e_n)\ \right]$. If $w \in \mathbb{R}^n$ is any vector then

$$T(w) = T\left(\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}\right) = T\left(w_1 \overset{e_1}{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} + w_2 \overset{e_2}{\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}} + \cdots + w_n \overset{e_n}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}\right)$$

by linearity

$$= w_1 T(e_1) + \cdots + w_n T(e_n)$$

by defn of matrix-vector mult.

$$= Aw.$$

Thus A is one matrix such that $T(v) = Av$ for all vectors $v \in \mathbb{R}^n$.

$T(v) = Av$ for all $v \in \mathbb{R}^n$

The theorem says that A is the the only matrix with this property.

To show this, suppose B is a $m \times n$ matrix with $T(v) = Bv$ for all $v \in \mathbb{R}^n$.

Then $\boxed{T(e_i) = Ae_i = Be_i}$ for all $i = 1, 2, \ldots, n$.

But Ae_i and Be_i are the ith columns of A and B.

Therefore A and B have the same columns, so they are the same matrix: $A = B$. $\square$

as more generally, we are assuming $T(v) = Av = Bv$ for all $v \in \mathbb{R}^n$

$m \times n$, $A = [T(e_1)\ T(e_2)\ \dots\ T(e_n)]$

We call A the $\boxed{\textit{standard matrix}}$ of the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$.

The standard matrix A is the unique matrix such that $T(v) = Av$ for all v.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the function $\boxed{T(v) = 3v.}$

This is a linear transformation, and its standard matrix has the formula

$$A = \left[\begin{array}{cccc} T(e_1) & T(e_2) & \dots & T(e_n) \end{array}\right] = \left[\begin{array}{cccc} 3e_1 & 3e_2 & \dots & 3e_n \end{array}\right] = \begin{bmatrix} 3 & 0 & \dots & 0 \\ 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 3 \end{bmatrix}.$$

A has nonzero entries only in positions $(1,1), (2,2), \dots, (n,n)$.

One calls such a matrix *diagonal*.

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}$ is the function

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1^2 + v_2^2 + \cdots + v_n^2.$$

This function is $\boxed{\textbf{not}}$ linear: we have $\boxed{T(2v) = 4T(v) \neq 2T(v)}$ for any $0 \neq v \in \mathbb{R}^n$.

$T(cv) \neq cT(v)$

Example. Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is the function

$$T\left(\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}\right) = \begin{bmatrix} v_n \\ \vdots \\ v_2 \\ v_1 \end{bmatrix}.$$

This function is a linear transformation. (Why?) Its standard matrix is

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \dots & T(e_{n-1}) & T(e_n) \end{bmatrix} = \begin{bmatrix} e_n & e_{n-1} & \dots & e_2 & e_1 \end{bmatrix} = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}.$$

In the matrix on the right, we adopt the convention of only writing the nonzero entries: all positions in the matrix which are blank contain zero entries.