MATH 2121 – Lecture # 8

Outline:

1. Review: matrix operations
2. Discuss invertible functions = [one to one + onto]
3. Invertible matrices

T has standard matrix A which is s x r

U has standard matrix B which is m x n

1 Last time: adding and multiplying matrices

Suppose $\boxed{T : \mathbb{R}^r \to \mathbb{R}^s}$ and $\boxed{U : \mathbb{R}^n \to \mathbb{R}^m}$ are linear.

Let A and B be the standard matrices with $\boxed{T(x) = Ax}$ and $\boxed{U(x) = Bx.}$

we can rescale T or A

1. $cT : \mathbb{R}^r \to \mathbb{R}^s$ for $c \in \mathbb{R}$ is the linear transformation with $\boxed{(cT)(x) = cT(x).}$

   Standard matrix of cT is matrix $\boxed{cA}$ formed by multiplying all entries by c.

   Definition. cA is the matrix $\begin{bmatrix} ca_1 & ca_2 & \dots & ca_r \end{bmatrix}$ where $A = \begin{bmatrix} a_1 & a_2 & \dots & a_r \end{bmatrix}$.

can add T+U or A+B

2. If $r = n$ and $s = m$ so A and B have the same size, then can form $T + U :$ $\mathbb{R}^n \to \mathbb{R}^m$ as the linear transformation with $\boxed{(T + U)(x) = T(x) + U(x).}$

   The standard matrix of $T + U$ is $\boxed{A + B}$ where:

   Definition. $A + B$ is formed by summing corresponding entries of A and B.

Ex $5\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 10 & 15 \end{bmatrix}$ Ex $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 4 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 4 \\ 0 & 1 & 1 \end{bmatrix}$

in the case when (codomain U) = (domain T)

rows of B = # columns of A

We can compose $T \circ U$ or multiply AB

3. If the domain of T is the codomain of U, meaning that $r = m$, then the composition $T \circ U : \mathbb{R}^n \to \mathbb{R}^s$ is the function $(T \circ U)(x) = T(U(x))$.

linear as T and U are linear

The standard matrix of $T \circ U$ is the product AB where:

Definition. The product AB is the matrix $AB = \begin{bmatrix} Ab_1 & Ab_2 & \dots & Ab_n \end{bmatrix}$ where $B = \begin{bmatrix} b_1 & b_2 & \dots & b_n \end{bmatrix}$.

Some remarks.

$A + B$ is only defined if A and B are matrices of the same size.

We have $A + B = B + A$ when either side is defined.

AB only defined if number of columns of A equals number of rows of B.

AB has same number of rows as A, and same number of columns as B.

Even if AB and BA are both defined, we may still have $AB \neq BA$.

$$\begin{bmatrix} 3 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \end{bmatrix}$$

Ex

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \left[\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 4 & 1 \\ 10 & 3 \\ 16 & 5 \end{bmatrix}$$

Ex

$$\begin{array}{cc} A & B \\ \left[\begin{array}{cc|cc} 1 & 2 & 3 & 5 \\ 6 & 7 & 8 & 9 \\ \hline 0 & 1 & 2 & 1 \\ 3 & 3 & 4 & 4 \end{array}\right] & \\ C & D \end{array} \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}}_{= \begin{bmatrix} O_{2\times 2} & I_2 \\ I_2 & O_{2\times 2} \end{bmatrix}} = \begin{bmatrix} \overbrace{AO+BI}^{=B} & \overbrace{AI+BO}^{=A} \\ \overbrace{CO+DI}^{=D} & \overbrace{CI+DO}^{=C} \end{bmatrix} = \begin{bmatrix} 3 & 5 & 1 & 2 \\ 8 & 9 & 6 & 7 \\ 2 & 1 & 0 & 1 \\ 4 & 4 & 3 & 3 \end{bmatrix}$$

$$O_{2\times 2} \overset{\text{def}}{=} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$$I_2 \overset{\text{def}}{=} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Fact If M is a 2×2 matrix then

$$MI_2 = M \quad \text{and} \quad MO_{2\times 2} = O_{2\times 2}$$

Remarks about block matrices:

Suppose A, B, C, D and W, X, Y, Z are matrices

such that can form $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $\begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$

This means $\begin{cases} \#\text{rows of } A = \#\text{rows of } B \\ \#\text{cols of } A = \#\text{cols of } C \\ \#\text{cols of } B = \#\text{cols of } D \\ \#\text{rows of } C = \#\text{rows of } D \end{cases}$ and likewise for W, X, Y, Z

If $\#\text{cols}(A) = \#\text{rows}(W)$
$\#\text{cols}(B) = \#\text{rows}(Y)$ then $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} AW+BY & AX+BZ \\ CW+DY & CX+DZ \end{bmatrix}$

row operations on a matrix A $\Big\} \longleftrightarrow \Big\{$ multiplying A on the left by an elementary matrix E to form EA

Example. Let $A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$ be a 3×4 matrix.

Consider what happens when we multiply A on the left by various 3×3 matrices.

swap rows :

1. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A = \begin{bmatrix} a & b & c & d \\ i & j & k & l \\ e & f & g & h \end{bmatrix}$: swaps rows 2 and 3 of A.

rescale a row :

2. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} a & b & c & d \\ 3e & 3f & 3g & 3h \\ i & j & k & l \end{bmatrix}$: rescales row 2 by factor 3.

row replacement :

3. $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A = \begin{bmatrix} a+2i & b+2j & c+2k & d+2l \\ e & f & g & h \\ i & j & k & l \end{bmatrix}$: adds $2 \times$ row3 to row1.

elementary matrix = a matrix obtained by doing one row op to $I = \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

Each of the matrices E that are left-multiplied with A are examples of *elementary matrices*: matrices formed by doing exactly one row operation to the *identity matrix*

$$I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \xrightarrow[\text{row op}]{\text{one}} E$$

that has all 1's on the diagonal and all 0's in off-diagonal positions.

If E is elementary, then EA is obtained from A by doing one row operation.

Important deduction: **row operations on a matrix correspond to multiplying on the left by elementary matrices.**

3 types of elementary matrices:

① permutations

② diagonal

$x \neq 0$

③ unipotent

$x \in \mathbb{R}$

$A^\top$ "A transpose"

2 Matrix transpose

The *transpose* of $m \times n$ matrix A is $n \times m$ matrix $A^\top$ formed by flipping A across the main diagonal, in order to interchange rows and columns.

$$\text{If } A = \cancel{\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}} \text{ and } \boxed{A^\top = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}}.$$

$$\text{If } C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 5 & -2 & 7 \\ 0 & 0 & 1 & 0 \end{bmatrix} \text{ then } C^\top = \begin{bmatrix} 1 & -3 & 0 \\ 1 & 5 & 0 \\ 1 & -2 & 1 \\ 1 & 7 & 0 \end{bmatrix}.$$

- $(A^\top)^\top = A$ since flipping twice does nothing.
- When defined, we have $(A + B)^\top = A^\top + B^\top$ and $(cA)^\top = c(A^\top)$ for all $c \in \mathbb{R}$
- When defined, we have $\boxed{(AB)^\top = B^\top A^\top}$.

order of multiplication is reversed

3 Inverses

Let $f : X \to Y$ be a function with domain X and codomain Y.

Definition. The function f is *invertible* or *bijective* if f is both onto and one-to-one.

Here is a more direct definition of an invertible function:

Proposition. The function f is invertible if and only if for each $b \in Y$ there is exactly one $a \in X$ with $f(a) = b$.

Proof. f is onto iff for each $b \in Y$ there is at least one $a \in X$ with $f(a) = b$.

f is one-to-one iff for each $b \in Y$ there is at most one $a \in X$ with $f(a) = b$.

Therefore f is both onto and one-to-one if and only if the given condition holds. $\square$

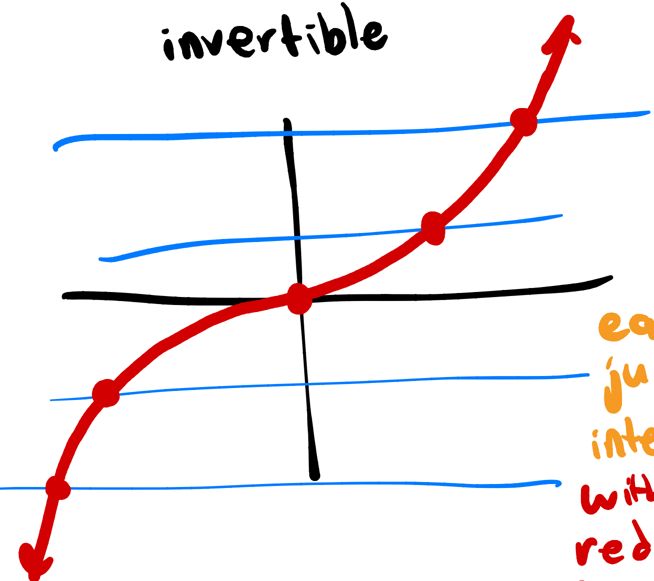
each elem of Y is the output of one and only one input in X (under f)

Calculus examples: $f: \mathbb{R} \to \mathbb{R}$ is invertible iff

each horizontal line intersects graph exactly once

$f(x) = x^3$

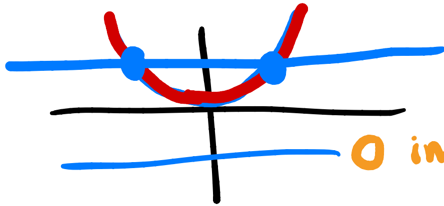
invertible



each blue line just one intersection with red graph

NOT invertible

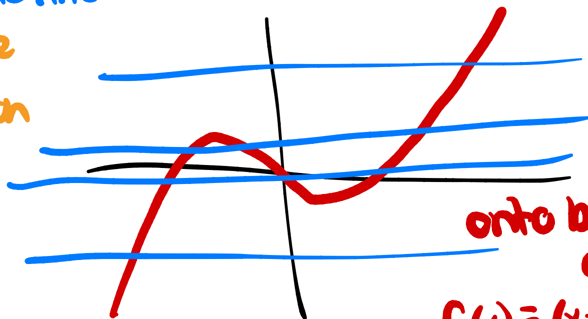
$f(x) = x^2$



2 intersections

0 intersections

neither onto, nor one-to-one



multiple intersections

onto but not one to one

$f(x) = (x-1)x(x+1)$

$f \circ id_X = f$ as $f \circ id_X(a) = f(id_X(a)) = f(a)$

$id_Y \circ f = f$ by similar reasoning

The *identity function* $\boxed{\mathrm{id}_X : X \to X}$ on a set X has formula $\boxed{\mathrm{id}_X(a) = a}$ for all $a \in X$.

If $f : X \to Y$ and $g : Y \to X$ are any functions then $f \circ \mathrm{id}_X = f$ and $\mathrm{id}_X \circ g = g$.

Example. The identity function on $\mathbb{R}^n$ is the linear transformation $\boxed{\mathrm{id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n}$ whose standard matrix is the $n \times n$ *identity matrix*

$$[e_1\ e_2 \cdots e_n] = I_n = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

standard matrix

If AI_n is defined then $\boxed{AI_n = A.}$ If I_nA is defined then $\boxed{I_nA = A.}$

often write I instead of I_n

Then always true that $AI = IA = A$

A function is invertible if and only if it has an *inverse* in the following sense.

Proposition. The function $f : X \to Y$ is invertible if and only if there is a function

$$f^{-1} : Y \to X$$

"f inverse"

such that $\boxed{f \circ f^{-1} = \mathrm{id}_Y \text{ and } f^{-1} \circ f = \mathrm{id}_X.}$

If there exists such a function f^{-1}, then it is unique, and we call it the *inverse* of f.

Ex $f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^3$

has inverse

$f^{-1}(x) = x^{1/3} = \sqrt[3]{x}$

Ex what about $g : \mathbb{R} \to \mathbb{R}$

not invertible

$g(x) = x^2$

why isn't the inverse of g given by $g^{-1}(x) = \sqrt{x}$?

the domain of $\sqrt{x}$ is not $\mathbb{R}$

How to detect invertibility of linear functions and compute the inverse?

Example. Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear function $\boxed{T(v) = Av \text{ for } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}$.

We have $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathsf{RREF}(A) = I$.

This means A has a pivot position in every row and every column.

This implies that T is onto and one-to-one, i.e., bijective.

(invertible)

What is the inverse T^{-1} of T?

We know that $T^{-1}: \mathbb{R}^2 \to \mathbb{R}^2$ exists.

Computing any single value $T^{-1}(y)$ amounts to solving $Ax = y$

Note that $T^{-1}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ is the unique vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ such that $\boxed{Ax = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$.

We can solve for x by row reducing the augmented matrix of this matrix equation:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & -2 & -3 \end{bmatrix} \sim \left[\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 3/2 \end{array}\right],$$

which means that the equation's unique solution is $x = \boxed{T^{-1}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 3/2 \end{bmatrix}}$

why?

$$A\underbrace{T^{-1}(y)}_{\text{treat this as variable } x} = T(T^{-1}(y)) = (T \circ T^{-1})(y) = id_{\mathbb{R}^2}(y) = y$$

treat this as variable x

then solve $Ax = y$

$$\text{Similarly, } T^{-1}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \text{ is the unique vector } x = \begin{bmatrix}x_1\\x_2\end{bmatrix} \text{ such that } \boxed{Ax = \begin{bmatrix}0\\1\end{bmatrix}}.$$

We again solve by row reduction to reduced echelon form:

$$[A \mid e_2] = \begin{bmatrix}1 & 2 & 0\\3 & 4 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 0\\0 & -2 & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 1\\0 & -2 & 1\end{bmatrix} \sim \left[\begin{array}{cc|c}1 & 0 & 1\\0 & 1 & -1/2\end{array}\right] = \text{RREF}$$

$$\text{which means that } y = \boxed{T^{-1}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\-1/2\end{bmatrix}}$$

Note: we have computed $T^{-1}(e_1)$ and $T^{-1}(e_2)$

so if T^{-1} were linear then its standard matrix

would be precisely $\left[T^{-1}(e_1)\;\; T^{-1}(e_2)\right] = \begin{bmatrix}-2 & 1\\3/2 & -1/2\end{bmatrix}$

If we knew that T^{-1} were linear, then we could conclude that

$$\boxed{T^{-1}(v) = Bv \quad \text{for} \quad B = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}}.$$

This turns out to be the right formula for T^{-1}. To see why, just check that

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

so $\boxed{T \circ T^{-1} = T^{-1} \circ T = \mathrm{id}_{\mathbb{R}^2}}.$

$T \circ T^{-1}(x) = T(T^{-1}(x)) = A(Bx) = (AB)x = Ix = x$

Similarly $T^{-1} \circ T(x) = B(Ax) = (BA)x = Ix = x$

It turns out that the inverse of an invertible linear transformation is always linear. Moreover, a linear transformation is invertible only if its standard matrix is square:

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a function.

Proposition. If T is linear and invertible, then $n = m$ and T^{-1} is linear.

Proof. From results last time, we know that T is onto only if $n \geq m$.

Likewise, T is one-to-one only if $n \leq m$.

If T is both onto and one-to-one, then necessarily $n = m$.

To check that T^{-1} is linear, use its definition to verify the two needed properties. $\square$

If T is linear then so is T^{-1} (when defined)

what is standard of T^{-1} in terms of T?

Definition. Let A be an $n \times n$ matrix and define $T : \mathbb{R}^n \to \mathbb{R}^n$ by $\boxed{T(x) = Ax.}$

The matrix A is *invertible* if the function T is invertible.

The *inverse* of A is the unique matrix A^{-1} such that $T^{-1}(x) = A^{-1}x$.

More concretely: A^{-1} = (standard matrix of T^{-1})

Proposition. Let A be an $n \times n$ matrix. The following mean the same thing:

1. A is invertible. call this the inverse of A
2. There is an $n \times n$ matrix A^{-1} such that $AA^{-1} = A^{-1}A = I_n$.
3. For each $b \in \mathbb{R}^n$ the equation $Ax = b$ has a unique solution.
4. $\text{RREF}(A) = I_n$.

A synonym for an invertible matrix is a *non-singular matrix*.

Reason for 4: this only way to have pivots in all rows and all columns

By def only square matrices can be invertible

Example (Warning).

Suppose $A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Then $\boxed{AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.}$

Neither A nor B is invertible, however.

There is no matrix A' such that $A'A = I_4$ and no matrix B' such that $BB' = I_3$.

A is the standard matrix of a linear transformation which is not one-to-one.

B is the standard matrix of a linear transformation which is not onto.

$$BA = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix} \neq I_4$$

Ex the 1×1 matrix $A = [a]$ is invertible iff $a \neq 0$
and then $A^{-1} = [1/a]$

The following is a useful formula for small calculations.

Theorem. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary 2×2 matrix.

(1) A is invertible if and only if $ad - bc \neq 0$.

(2) If $ad - bc \neq 0$ then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Proof. When $ad - bc = 0$, can deduce that one row is a scalar multiple of the other, so $\mathsf{RREF}(A)$ has at least one zero row and is not equal to I_2, so A is not invertible.

When $ad - bc$, can check directly that formula for A^{-1} has $AA^{-1} = A^{-1}A = I_2$. $\square$

Example. Earlier example showed that $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$.

$A^{-1} =$ ① swap diag entries
② negate offdiag entries
③ divide by det $= ad - bc$

$$A^{-1}A = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} da-bc & db-bd \\ -ca+ac & -bc+ad \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (\text{also } AA^{-1})$$

$A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$

$(A^{-1})^T A^T = (AA^{-1})^T = I^T = I$

like $\frac{1}{1/a} = a$ if $0 \neq a \in \mathbb{R}$

Theorem. Let A and B be $n \times n$ matrices.

1. If A is invertible then $\boxed{(A^{-1})^{-1} = A.}$
2. If A and B are both invertible then AB is invertible with $\boxed{(AB)^{-1} = B^{-1}A^{-1}.}$
3. If A is invertible then $A^\top$ is invertible with $\boxed{(A^\top)^{-1} = (A^{-1})^\top.}$

A corollary of this theorem is that the product of a list of $n \times n$ invertible matrices is itself invertible, with inverse the product of the inverses in reverse order. In symbols:

$$\boxed{(ABC \cdots Z)^{-1} = Z^{-1} \cdots C^{-1}B^{-1}A^{-1}.}$$

$(AB)(B^{-1}A^{-1})$

$= \cdots$

$= I$

and

$(B^{-1}A^{-1})(AB)$

$= B^{-1}(A^{-1}A)B$

$= B^{-1}IB = B^{-1}B = I$

How to compute A^{-1} if A is $n \times n$ and $n > 2$?

Checking whether a matrix is invertible is almost same process as computing inverse.

Process to compute A^{-1}

1 Let A be an $n \times n$ matrix. Consider the $n \times 2n$ matrix $\begin{bmatrix} A & I_n \end{bmatrix}$. $= \left[\begin{array}{c|ccc} A & 1 & & 0 \\ & & \ddots & \\ & 0 & & 1 \end{array}\right]$

2 If A is invertible then $\boxed{\text{RREF}\left(\begin{bmatrix} A & I_n \end{bmatrix}\right) = \begin{bmatrix} I_n & A^{-1} \end{bmatrix}}$

So to compute A^{-1}, row reduce $\begin{bmatrix} A & I_n \end{bmatrix}$, and then take the last n columns.

in general, for any square matrix A,

$$\text{RREF}([A \ I]) = [\text{RREF}(A) \ B]$$

if A is invertible, $\text{RREF}(A) = I$ and $A^{-1} = B$

(otherwise $\text{RREF}(A) \neq I$)

form [A I]

Example. To find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ we row reduce

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right].$$

Now check directly that $A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$