MATH 2121 – Lecture #9

Plan today:

- some review of inverses & inverse matrix
- subspaces, Col A, Nul A, basis

1 Last time: inverses

The following all mean the same thing for a function $f : X \to Y$:

1. f is *invertible*.
2. f is one-to-one and onto.
3. For each $b \in Y$ there is exactly one $a \in X$ with $f(a) = b$.
4. There is a unique function $f^{-1} : Y \to X$, called the *inverse* of f, such that

must be able to compose in both orders and get identity function

$$f^{-1}(f(a)) = a \quad \text{and} \quad f(f^{-1}(b)) = b \quad \text{for all } a \in X \text{ and } b \in Y.$$

Proposition. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear and invertible then $m = n$ and T^{-1} is linear.

Recall: A is standard matrix of T if $T(v) = Av$ for all v in domain of T

The following all mean the same thing for an $n \times n$ matrix A:

1. A is $\boxed{\textit{invertible}.}$
2. A is the standard matrix of an invertible linear function $T : \mathbb{R}^n \to \mathbb{R}^n$.
3. There is a unique $n \times n$ matrix A^{-1}, called the $\boxed{\textit{inverse}}$ of A, such that

two conditions

$$\left\{ A^{-1}A = AA^{-1} = I_n \right. \qquad \text{where we define } I_n = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = I$$

4. For each $b \in \mathbb{R}^n$ the equation $Ax = b$ has a unique solution.
5. $\boxed{\mathsf{RREF}(A) = I_n}$ ← computable
6. The columns of A are $\underbrace{\text{linearly independent}}_{Ax=b \text{ has } \leq 1 \text{ solution}}$ and their $\underbrace{\text{span}}_{Ax=b \text{ has } \geq 1 \text{ solution}}$ is $\mathbb{R}^n$.

Warm-up: inverse of a 1×1 matrix $A=[a]$ exists when $a\neq 0$ and is $A^{-1}=[1/a]$

Proposition. Let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix.

(1) If $\boxed{ad-bc=0}$ then A is not invertible.

(2) If $\boxed{ad-bc\neq 0}$ then $\boxed{A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$.

swap diag
negate off diag
divide by det

Proposition. Let A and B be $n\times n$ matrices.

1. If A is invertible then $(A^{-1})^{-1}=A$.
2. If A and B are both invertible then AB is invertible and $\boxed{(AB)^{-1}=B^{-1}A^{-1}}$.
3. If A is invertible then A^T is invertible and $\boxed{(A^T)^{-1}=(A^{-1})^T}$.

hence also square

use $(AB)^T=B^TA^T$

just check

$MM^{-1}=M^{-1}M=I$

use this when A is nxn
with n>2

Process to compute A^{-1}

0 Let A be an $n \times n$ matrix. Consider the $n \times 2n$ matrix $\begin{bmatrix} A & I_n \end{bmatrix}$.

2 If A is invertible then $\mathsf{RREF}\left(\begin{bmatrix} A & I_n \end{bmatrix}\right) = \begin{bmatrix} I_n & A^{-1} \end{bmatrix}$.

So to compute A^{-1}, row reduce $\begin{bmatrix} A & I_n \end{bmatrix}$ to reduced echelon form.

Then take the last n columns.

In general RREF([A I]) = [RREF(A) B]

If A is not invertible then RREF(A) ≠ I
but otherwise B gives you A^{-1}.

Ex Let's invert $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 4 & 0 & 5 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & -3 & -4 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{4}{3} & 0 & -\frac{1}{3} \end{array}\right]$$

$$\to \Bigg[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{=I_3} \Bigg| \underbrace{\begin{matrix} -5/3 & 0 & 2/3 \\ 0 & 1/3 & 0 \\ 4/3 & 0 & -1/3 \end{matrix}}_{=A^{-1}}\Bigg] \Rightarrow A^{-1} = -\frac{1}{3}\begin{bmatrix} 5 & 0 & -2 \\ 0 & -1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

Now check $AA^{-1} = A^{-1}A = I_3 \ldots$

in general situations when we can multiply but X * Y ≠ Y * X (using some operation *) you can have X * Y = 1 but Y * X ≠ 1

2 Stronger characterization of invertible matrices

but for n x n matrices this does not happen!

A matrix can only be invertible if it has the same number of rows and columns.

Theorem. When A is a square $n \times n$ matrix, the following are equivalent:

(a) A is invertible.

(b) The columns of A are linearly independent.

(c) The span of the columns of A is $\mathbb{R}^n$

before we only claimed (a) $\Leftarrow$ ((b) & (c))

Corollary. Suppose A and B are both $n \times n$ matrices. If $AB = I_n$ then $BA = I_n$.

So when A and B are both square, to show that $B = A^{-1}$, can just check $AB = I_n$.

Important note: this principle only applies to *square matrices.*

so actually you can just do one multiplication to check if B is A^{-1} when A and B are square

Cautionary example for non-square matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

but neither matrix is square or invertible

observe: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq I_3$

informally: subsets given by $\mathbb{R}$-span{some vectors}

3 $\boxed{\text{Subspaces}}$ of $\mathbb{R}^n$

zero vector

Let n be a positive integer. Remember that $0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$.

Definition. Let H be a subset of $\mathbb{R}^n$. The subset H is a *subspace* if

1. $0 \in H$, → subspaces are nonempty
2. $u + v \in H$ for all $u, v \in H$, and → subspaces are closed under +
3. $cv \in H$ for all $c \in \mathbb{R}$ and $v \in H$. → subspaces are closed under scalar multiplication

a subspace is a nonempty subset of $\mathbb{R}^n$ closed under linear combinations

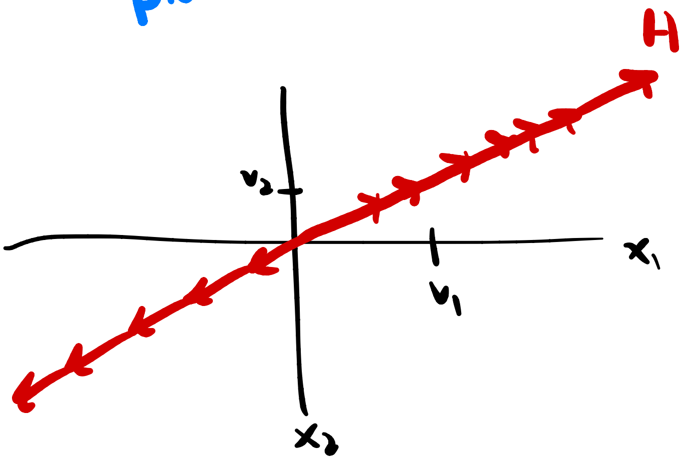
Ex The only subspaces of $\mathbb{R}^1 = \mathbb{R}$ are $\{0\}$ and $\mathbb{R}^1$

Pf If H is a subspace of $\mathbb{R}^1$ and $0 \neq x \in H$ then $cx \in H$ for all $c \in \mathbb{R}$ which means $y \in H$ for all $y \in \mathbb{R}^1$

Ex Subspaces of $\mathbb{R}^2$ include $\{0\} = \left\{\begin{bmatrix}0\\0\end{bmatrix}\right\}$, $\mathbb{R}^2$ itself,

and $\mathbb{R}\text{-span}\left\{\begin{bmatrix}v_1\\v_2\end{bmatrix} \neq \begin{bmatrix}0\\0\end{bmatrix}\right\} = H$ (any line through origin)

picture of this:



$av + bv = (a+b)v$

for $a, b \in \mathbb{R}$

call $\{0\}$ the zero subspace

Common examples

$\mathbb{R}^n$ is a subspace of itself. (largest subspace)

The set $\{0\}$ consisting of just the zero vector is a subspace of $\mathbb{R}^n$. (smallest subspace)

The empty set $\varnothing$ is *not* a subspace since it does not contain the zero vector.

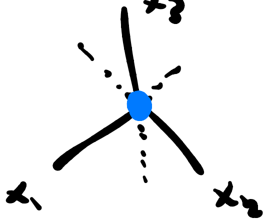
If $H \subseteq \mathbb{R}^2$ is a subspace then $H = \{0\}$, $H = \mathbb{R}^2$, or $H = \mathbb{R}\text{-span}\{v\}$ for some $v \in \mathbb{R}^2$

The span of any set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

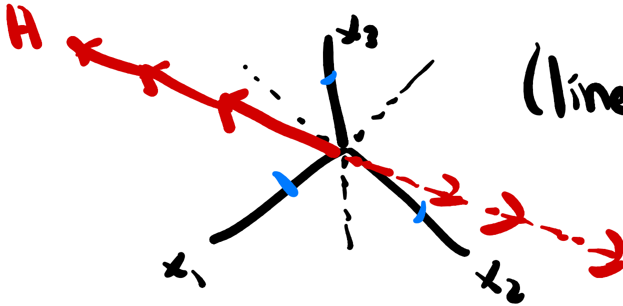
Conversely, will see that any subspace of $\mathbb{R}^n$ is the span of a finite set of vectors.

Ex Subspaces of $\mathbb{R}^3$

$\{0\}$

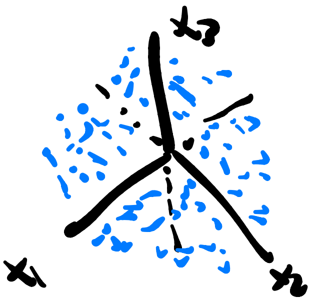


$H = \mathbb{R}\text{-span}\left[\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \neq 0\right]$

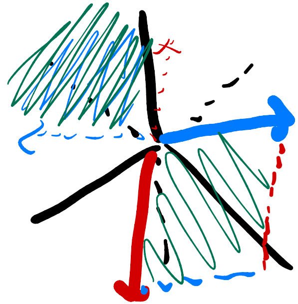


(line)

$\mathbb{R}^3$



$H = \mathbb{R}\text{-span}[u, v]$



(plane)

$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \notin X$

so not a subspace

Example. Not a subspace (does not contain zero vector):

$$X = \left\{ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 1 \right\}.$$

Example. The set

$$H = \left\{ v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 : v_1 + v_2 + v_3 = 0 \right\}$$

is a subspace since if $u, v \in H$ and $c \in \mathbb{R}$ then

both zero as $u, v \in H$

$$(u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) = \overbrace{(u_1 + u_2 + u_3)} + \overbrace{(v_1 + v_2 + v_3)} = 0 + 0 = 0$$

and

$$cv_1 + cv_2 + cv_3 = c\overbrace{(v_1 + v_2 + v_3)} = 0$$

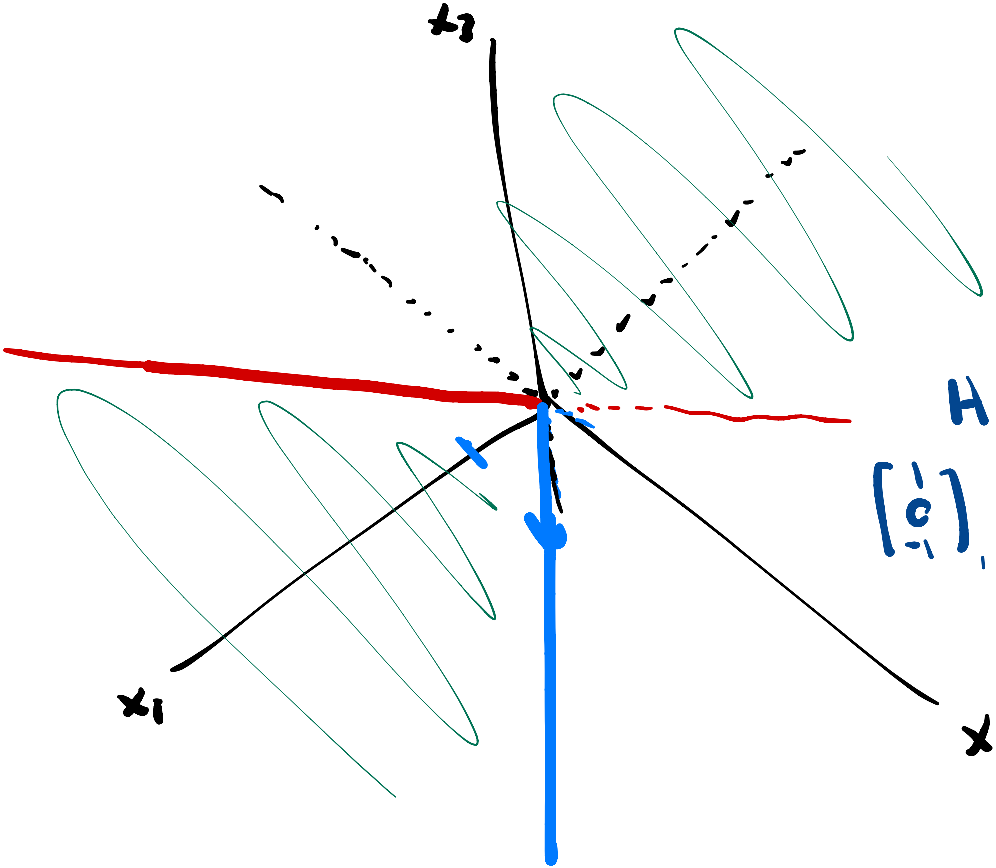
so $u + v \in H$ and $cv \in H$.

this is a subspace, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

(not obvious)

What does $H = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; v_1 + v_2 + v_3 = 0 \right\}$ look like?

it's a **plane**



H contains

$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ etc.

Any matrix A gives rise to two subspaces, called the *column space* and *null space*.

Definition. The *column space* of an $m \times n$ matrix A is the subspace

$$\operatorname{Col} A = \{Ax : x \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

The set $\operatorname{Col} A$ is the span of the columns of A.

Example. If $V = \mathbb{R}\text{-span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{u}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{w} \right\}$ then what matrices A have $\boxed{\operatorname{Col} A = V}$? ?

u+2w

2u+w

Here are four examples:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 & 1 & 2 \end{bmatrix}.$$

Many different matrices can have the same column space, and it may not be at all obvious whether a subspace V is equal to the column space of a given matrix A.

$\underline{\text{Ex}}$ $H = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \middle| \ v_1 + v_2 + v_3 = 0 \right\} = \text{Col } A$

for $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix}$

also for $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ -1 & -1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix}$ etc.

Col → apply to a matrix A

range → apply to a function T

Remark. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is the linear function $T(x) = Ax$ then $\boxed{\mathrm{Col}\, A = \mathsf{range}(T).}$

A vector $b \in \mathbb{R}^m$ belongs to $\mathrm{Col}\, A$ if and and only if $Ax = b$ has a solution.

Thus $\mathrm{Col}\, A = \mathbb{R}^m$ if and only if $Ax = b$ has a solution for each $b \in \mathbb{R}^m$.

Definition. The *null space* of an $m \times n$ matrix A is the subspace

$$\mathrm{Nul}\, A = \{v \in \mathbb{R}^n : Av = 0\} \subseteq \mathbb{R}^n$$

The set $\mathrm{Nul}\, A$ is exactly the set of solutions to the matrix equation $Ax = 0$.

Proof that $\mathrm{Nul}\, A$ *is a subspace.* We have $0 \in \mathrm{Nul}\, A$. Let $u, v \in \mathrm{Nul}\, A$ and $c \in \mathbb{R}$.

Then $A(u + v) = Au + Av = 0 + 0 = 0$ and $A(cv) = c(Av) = 0$.

So $u + v \in \mathrm{Nul}\, A$ and $cv \in \mathrm{Nul}\, A$. Thus $\mathrm{Nul}\, A$ is a subspace of $\mathbb{R}^n$. □

Note: if A is m×n the $\mathrm{Col}\, A \subseteq \mathbb{R}^m$

$\mathrm{Nul}\, A \subseteq \mathbb{R}^n$

<u>Ex</u> Notice $H = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \middle| \ v_1 + v_2 + v_3 = 0 \right\} = \mathrm{Nul}\, B$

for $B = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ $\left(\text{or } B = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \text{ etc} \right)$

since $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 + v_2 + v_3$

Remark. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear with $T(x) = Ax$ then

$$\operatorname{Nul} A = \{x \in \mathbb{R}^n : T(x) = 0\}.$$

The column space is a subspace of $\mathbb{R}^m$ **where** m **is the number of rows of** A.

The null space is a subspace of $\mathbb{R}^n$ **where** n **is the number of columns of** A.

A subspace can be completely determined by a finite amount of data.

This data will be called a *basis*.

plural of "basis" = "bases"

Definition. Let H be a subspace of $\mathbb{R}^n$. A set of vectors $v_1, v_2, \ldots, v_k \in H$ is a *basis* for H if the vectors are linearly independent and their span is equal to H.

The empty set $\varnothing$ is considered to be a basis for the zero subspace $\{0\}$.

Example. Remember the vectors $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$ where $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

These form a basis for $\mathbb{R}^n$. We call this the *standard basis* of $\mathbb{R}^n$.

Theorem. Every subspace H of $\mathbb{R}^n$ has a basis of size at most n.

Easy reason why a basis for $H \subseteq \mathbb{R}^n$ must have $\leq n$ elements:

$\boxed{\text{any set of more than } n \text{ vectors in } \mathbb{R}^n \text{ is linearly dependent}}$

Note: $H = \{0\} \subseteq \mathbb{R}^n$ but its basis has zero elements

Also: $H = \mathbb{R}\text{-span}\{e_1\} = \left\{ \begin{bmatrix} x \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \, x \in \mathbb{R} \right\} \subseteq \mathbb{R}^n$

has basis $\{e_1\}$ with just one element

(but $\{2e_1\}$ is another basis)

How to find a basis for Col A or Nul A?

Example. Let $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$.

Nul A = $\{x \in \mathbb{R}^5 \mid Ax = 0\}$

How can we find a basis for Nul A?

Finding a basis for Nul A is more or less the same task as finding all solutions to the homogeneous equation $\underline{Ax = 0}$.

So let's first try to solve that equation.

If we row reduce the 3×6 matrix $\begin{bmatrix} A & 0 \end{bmatrix}$, we get

$$\left[\begin{array}{c|c} A & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] = \mathsf{RREF}(\begin{bmatrix} A & 0 \end{bmatrix}).$$

pivots in 2 columns, not in last column

2 basic vars

3 free vars

same solutions as RREF(A)x = 0

This tells us that $\boxed{Ax = 0}$ if and only if

expresses basic vars in terms of free vars

RREF $\leftrightarrow$ $\begin{cases} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \end{cases}$ or equivalently $\begin{cases} x_1 = 2x_2 + x_4 - 3x_5 \\ x_3 = -2x_4 + 2x_5. \end{cases}$

By substituting these formulas, we deduce that $x \in \operatorname{Nul} A$ if and only if

Ax = 0 $\Rightarrow$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ are a basis for $\operatorname{Nul} A$.

This example is important: the procedure just described works to construct a basis of $\operatorname{Nul} A$ for any matrix A. **The size of this basis will always be equal to the number of free variables in the linear system $Ax = 0$.**

How to find a basis for $\operatorname{Nul} A$ is something you should learn and remember.

Example. Let $B = \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

What is a basis for Col B?

This matrix is in reduced echelon form. How to find a basis for $\operatorname{Col} B$?

The columns of B automatically span $\operatorname{Col} B$, but might be linearly dependent.

The largest linearly independent subset of the columns of B will be a basis for $\operatorname{Col} B$.

In our example, the pivot columns 1, 2 and 5 are linearly independent since each has a row with a 1 where the others have 0s.

These columns span columns 3 and 4, so a basis for $\operatorname{Col} B$ is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

This example was special since the matrix B was already in reduced echelon form. To find a basis of the column space of an arbitrary matrix, we use this observation:

Proposition. Let A be any matrix. The pivot columns of A form a basis for $\operatorname{Col} A$.

(not the only basis)

(recall: to find pivot columns, still need to compute RREF(A))

$$\mathrm{RREF}(A) = B$$

$$= \begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example. The matrix

$$\begin{array}{cc} & \begin{array}{ccccc} \text{column } 1 & 2 & & & 5 \end{array} \\ A = & \begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix} \end{array}$$

is row equivalent to the matrix B in the previous example.

Columns 1, 2, 5 have pivots, so $\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -9 \\ 2 \\ 1 \\ -8 \end{bmatrix} \right\}$ is a basis for $\mathrm{Col}\, A$.

Next time: we will show that if H is a subspace of $\mathbb{R}^n$ then all of its bases have the same size. The common size of each basis is the *dimension* of H.