MATH 2121 — Lecture #10

Outline today:

1. Review: subspace, column space, null space, basis

2. New things: dimension
   rank
   basis theorems

1 Last time: subspaces

contains zero vector

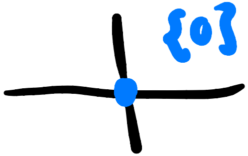
Definition. A subset H of $\mathbb{R}^n$ is a $\boxed{\textit{subspace}}$ if $0 \in H$, $\underline{u + v \in H}$, and $\underline{cv \in H}$ for all $u, v \in H$ and $c \in \mathbb{R}$.

A subspace is a nonempty set containing all linear combinations of its vectors.

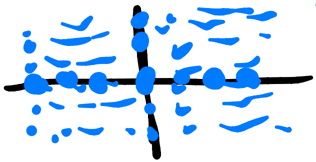
Example. Examples of subspaces of $\mathbb{R}^n$:

- The set $\boxed{\{0\}}$ containing just the zero vector and the set $\boxed{\mathbb{R}^n}$ itself. $\{0\}$ zero subspace
- The set of all scalar multiples of a single vector.
- The span of any set of vectors in $\mathbb{R}^n$.
- The range of a linear function $T : \mathbb{R}^k \to \mathbb{R}^n$. (column space)
- The set of vectors v with $T(v) = 0$ for a linear function $T : \mathbb{R}^n \to \mathbb{R}^k$. (null space)

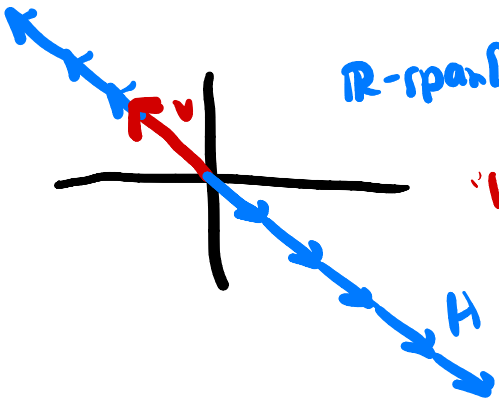
Ex subspaces of $\mathbb{R}^2$



$\{0\} = H$ "point"



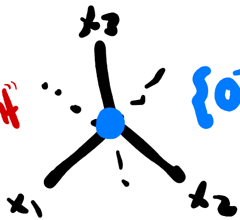
$\mathbb{R}^2 = H$ "plane"



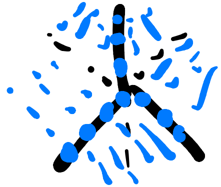
$\mathbb{R}\text{-span}\{v\} = H$

"line"

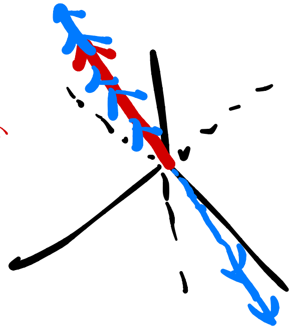
Ex subspaces of $\mathbb{R}^3$



"point" $\{0\}$

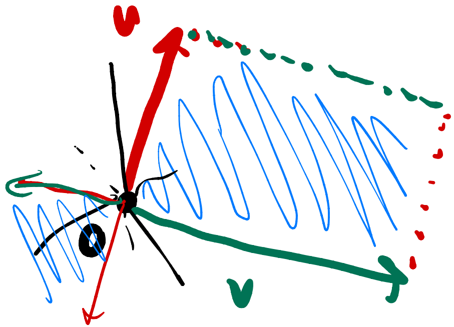


$\mathbb{R}^3$



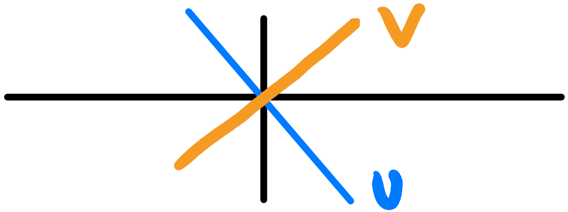
"line"

$\mathbb{R}\text{-span}\{v\}$



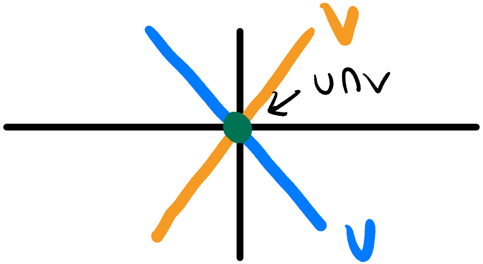
$\mathbb{R}\text{-span}\{u, v\}$

"plane"



$U \cup V$ not subspace, not closed under $+$

- The union of two subspaces is not necessarily a subspace. (Consider two lines in $\mathbb{R}^2$)
- The sum of two subspaces U and V is the set $U + V = \{u + v : u \in U \text{ and } v \in V\}$. This is a subspace.
- The intersection of two subspaces is always a subspace. (Check the conditions.)



$u, v \in U \Rightarrow u+v \in U$

$u, v \in V \Rightarrow u+v \in V$

$u, v \in U \cap V \Rightarrow u+v \in U \cap V$

$U \cap V = \{0\}$

Main ways to construct and analyze subspaces

(span of columns of matrix A) $\subseteq \mathbb{R}^{\#\text{rows of } A}$

Definition. To any $m \times n$ matrix A there are two corresponding subspaces:

1. The *column space* of A is the subspace $\boxed{\operatorname{Col} A} = \{Ax : x \in \mathbb{R}^n\}$ of $\mathbb{R}^m$.
2. The *null space* of A is the subspace $\boxed{\operatorname{Nul} A} = \{x \in \mathbb{R}^n : Ax = 0\}$ of $\mathbb{R}^n$.

It is not obvious from these definitions, but it will turn out that each subspace of $\mathbb{R}^m$ occurs as the column space of some matrix. Likewise, each subspace of $\mathbb{R}^n$ occurs as the null space of some matrix.

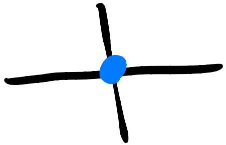
- If A and B have same number of rows then $\operatorname{Col}\begin{bmatrix} A & B \end{bmatrix} = \operatorname{Col} A + \operatorname{Col} B$.
- If A and B have same number of columns then $\operatorname{Nul}\begin{bmatrix} A \\ B \end{bmatrix} = \operatorname{Nul} A \cap \operatorname{Nul} B$.

(set of solutions x to $Ax = 0$) $\subseteq \mathbb{R}^{\#\text{columns of } A}$

why? for $x \in \mathbb{R}^n$ we have $\begin{bmatrix} A \\ B \end{bmatrix} x = \begin{bmatrix} Ax \\ Bx \end{bmatrix} \overset{?}{=} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$

iff $Ax = 0$ and $Bx = 0$

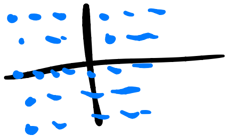
Ex How to see subspaces of $\mathbb{R}^2$ as $\text{Col}\,A$ or $\text{Nul}\,A$

① 

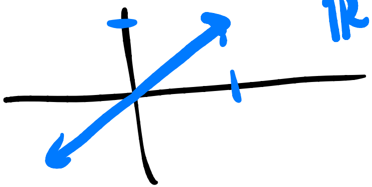
zero subspace $\left\{\begin{bmatrix}0\\0\end{bmatrix}\right\} = \text{Col}\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix} = \text{Nul}\begin{bmatrix}1&0\\0&1\end{bmatrix}$

since $\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$

is zero iff $\begin{bmatrix}v_1\\v_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

② 

$$\mathbb{R}^2 = \underbrace{\text{Col}\begin{bmatrix}1&0\\0&1\end{bmatrix}}_{=\mathbb{R}\text{-span}(e_1,e_2)=\mathbb{R}^2} = \text{Nul}\begin{bmatrix}0&0\\0&0\\0&0\end{bmatrix}$$

③ 

1x2 matrix

$$\mathbb{R}\text{-span}\left\{\begin{bmatrix}v_1\\v_2\end{bmatrix}\right\} = \text{Col}\begin{bmatrix}v_1\\v_2\end{bmatrix} = \text{Nul}\begin{bmatrix}-v_2 & v_1\end{bmatrix}$$

check:

$\rightarrow \begin{bmatrix}-v_2 & v_1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = -v_2x_1 + v_1x_2$

this is zero

iff $\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$

or if $\begin{bmatrix}x_1\\x_2\end{bmatrix} = c\begin{bmatrix}v_1\\v_2\end{bmatrix}$

Definition. A $\boxed{\textit{basis}}$ of a subspace H of $\mathbb{R}^n$ is a set of linearly independent vectors whose span is H.

An important basis with its own notation: the *standard basis* of $\mathbb{R}^n$.

This consists of the vectors $e_1, e_2, \ldots, e_n$ where e_i is the vector in $\mathbb{R}^n$ with 1 in row i and 0 in all other rows.

Theorem. Every subspace H of $\mathbb{R}^n$ has a basis of size at most n.

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Important algorithms to find bases

Let A be an $m \times n$ matrix.

How to find a basis of $\operatorname{Nul} A$. $\subseteq \mathbb{R}^n$

or $[A|0]$

1. Find all solutions to $Ax = 0$ by row reducing A to echelon form.

   Recall that x_i is a *basic variable* if column i of $\mathsf{RREF}(A)$ contains a leading 1, and that otherwise x_i is a *free variable*.

2. Express each basic variable in terms of the free variables, and then write

   by writing down the linear system
   $\mathrm{RREF}(A)x = 0$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \boxed{x_{i_1} b_1 + x_{i_2} b_2 + \cdots + x_{i_k} b_k}$$

   scalar   numeric vectors

   where $x_{i_1}, x_{i_2}, \ldots, x_{i_k}$ are the free variables and $b_1, b_2, \ldots, b_k \in \mathbb{R}^n$.

3. The vectors $b_1, b_2, \ldots, b_k$ then form a basis for $\operatorname{Nul} A$.

RREF(A)x = 0

basic: x_1, x_2

free: x_3, x_4

Example. Suppose $A = \begin{bmatrix} 1 & 2 & 5 & 8 \\ 2 & 3 & 7 & 0 \end{bmatrix}$. The RREF($A$) $= \begin{bmatrix} 1 & 0 & -1 & -24 \\ 0 & 1 & 3 & 16 \end{bmatrix}$.

1. So $Ax = 0$ if and only if $\boxed{\begin{cases} x_1 - x_3 - 24x_4 = 0 \\ x_2 + 3x_3 + 16x_4 = 0 \end{cases}}$ which means $\boxed{\begin{cases} x_1 = x_3 + 24x_4 \\ x_2 = -3x_3 - 16x_4. \end{cases}}$

2. Thus x_1, x_2 are basic variables, x_3, x_4 are free variables, and if $\underline{Ax = 0}$ then
$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 24x_4 \\ -3x_3 - 16x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 24 \\ -16 \\ 0 \\ 1 \end{bmatrix}.$$

3. The set of vectors $\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 24 \\ -16 \\ 0 \\ 1 \end{bmatrix} \right\}$ is then a basis for Nul A.

How to find a basis of Col A**.**

1. The pivot columns of A form a basis of Col A.

This looks simpler than the previous algorithm, but still have to compute $\mathsf{RREF}(A)$.

$$\text{RREF}(A) = \begin{bmatrix} 1 & 0 & -1 & -24 \\ 0 & 1 & 3 & 16 \end{bmatrix}$$

Example. If $A = \begin{bmatrix} 1 & 2 & 5 & 8 \\ 2 & 3 & 7 & 0 \end{bmatrix}$ then columns 1, 2 have pivots so

$$\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$

is a basis for Col A. $\subseteq \mathbb{R}^2$

This is not the only set of columns of A that forms a basis for Col A, however.

Goal: understand why every basis for a given subspace has the same # of elements

2 Coordinate systems

$x_1 b_1 + x_2 b_2 + \dots = v$ always has exactly one solution (if $v \in H$)

Suppose H is a subspace of $\mathbb{R}^n$. Let $b_1, b_2, \ldots, b_k$ be a basis of H.

Theorem. Let $v \in H$. There are unique coefficients $c_1, c_2, \ldots, c_k \in \mathbb{R}$ such that

$$c_1 b_1 + c_2 b_2 + \cdots + c_k b_k = v.$$

Let $\mathcal{B} = (b_1, b_2, \ldots, b_k)$ be the list containing our basis vectors in some fixed order.

For $v \in H$, let $\boxed{[v]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}}$ $\in \mathbb{R}^k$ be unique vector with $c_1 b_1 + c_2 b_2 + \cdots + c_k v_k = v$.

Equivalently, $[v]_{\mathcal{B}}$ is unique solution to matrix equation $\begin{bmatrix} b_1 & b_2 & \cdots & b_k \end{bmatrix} x = v$.

We call $[v]_{\mathcal{B}}$ the *coordinate vector of v in the basis $\mathcal{B}$* or just *v in the basis $\mathcal{B}$*.

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ v_2 \\ \vdots \\ 0 \end{bmatrix} + \cdots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ v_n \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots$$

$$= \sum_{i=1}^{n} v_i e_i$$

* **Example.** If $H = \mathbb{R}^n$ and $\mathcal{B} = (e_1, e_2, \ldots, e_n)$ is the standard basis then $[v]_{\mathcal{B}} = v$.

* **Example.** If $H = \mathbb{R}^n$ and $\mathcal{B} = (e_n, \ldots, e_2, e_1)$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then $[v]_{\mathcal{B}} = \begin{bmatrix} v_n \\ \vdots \\ v_2 \\ v_1 \end{bmatrix}$.

If $B = (e_1, 2e_2, 3e_3, 4e_4, \ldots, ne_n)$

then $[v]_B = \begin{bmatrix} v_1 \\ v_2/2 \\ v_3/3 \\ \vdots \\ v_n/n \end{bmatrix}$ because $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 e_1 + \frac{v_2}{\cancel{2}} \cancel{2} e_2$

$+ \frac{v_3}{\cancel{3}} \cancel{3} e_3 +$

$\ldots + \frac{v_n}{\cancel{n}} \cancel{n} e_n$

In general, to find $[v]_B$ it is necessary to solve the matrix equation

$$\begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix} = v$$

if $B = (b_1, b_2, \ldots, b_k)$

Example. Let $b_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$ and $b_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$.

What is $[v]_B$?

need to solve

Then $\mathcal{B} = (b_1, b_2)$ is a basis for $H = \mathbb{R}\text{-span}\{b_1, b_2\}$, which is a subspace of $\mathbb{R}^3$.

The unique $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ such that $\begin{bmatrix} 3 & -1 \\ 6 & 0 \\ 2 & 1 \end{bmatrix} x = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$ is found by row reduction:

$$\left[\begin{array}{cc|c} 3 & -1 & 3 \\ 6 & 0 & 12 \\ 2 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 3 & -1 & 3 \\ 2 & 1 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & -1 & -3 \\ 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right].$$

The last matrix implies that $x_1 = 2$ and $x_2 = 3$ so $[v]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

now check: $2\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$

$b_i = e_i - e_{i+1}$ $(i = 1, 2, \ldots, n-1)$

Example. If $b_1 = e_1 - e_2$, $b_2 = e_2 - e_3$, $b_3 = e_3 - e_4$, $\ldots$, $b_{n-1} = e_{n-1} - e_n$ and

$$v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ -v_1 - v_2 - \cdots - v_{n-1} \end{bmatrix}$$

then $v \in H = \mathbb{R}\text{-span}\{b_1, b_2, \ldots, b_{n-1}\}$ and

$$[v]_{\mathcal{B}} = \begin{bmatrix} v_1 \\ v_1 + v_2 \\ v_1 + v_2 + v_3 \\ v_1 + v_2 + v_3 + v_4 \\ \vdots \\ v_1 + v_2 + v_3 + \cdots + v_{n-1} \end{bmatrix} \in \mathbb{R}^{n-1}.$$

finding this requires some computation, (but checking is easy)

$[v]_B$ has # rows = # elems of B

Main result:

Theorem. Every basis of a subspace $H \subseteq \mathbb{R}^n$ has the same number of elements.

1

Proof. Suppose $\mathcal{B} = (b_1, b_2, \dots, b_k)$ and $\mathcal{B}' = (b'_1, b'_2, \dots, b'_l)$ are two bases of H. Assume $k \neq l$. We might as well assume $k < l$; if $k > l$ then switch $\mathcal{B}$ and $\mathcal{B}'$.

goal: derive a contradiction

2

Then $[b'_1]_{\mathcal{B}}, [b'_2]_{\mathcal{B}}, \dots, [b'_l]_{\mathcal{B}}$ are $l > k$ vectors $\boxed{\text{in } \mathbb{R}^k}$ so are linearly dependent. This means there exist coefficients $c_1, c_2, \dots, c_l \in \mathbb{R}$, not all zero, with

$$0 = c_1[b'_1]_{\mathcal{B}} + c_2[b'_2]_{\mathcal{B}} + \cdots + c_l[b'_l]_{\mathcal{B}} = [c_1 b'_1 + c_2 b'_2 + \cdots + c_l b'_l]_{\mathcal{B}}.$$

3 $x \mapsto [x]_B$ is LINEAR as function of x

4

This means that

$$c_1 b'_1 + c_2 b'_2 + \cdots + c_l b'_l \overset{\text{def}}{=} \begin{bmatrix} b_1 & b_2 & \cdots & b_k \end{bmatrix} \underbrace{[c_1 b'_1 + c_2 b'_2 + \cdots + c_l b'_l]_{\mathcal{B}}}_{=0 \text{ by prev step}} = 0.$$

Since the coefficients c_i are not all zero, this contradicts the fact that $b'_1, b'_2, \dots, b'_l$ are linearly independent. This means we cannot actually have $k \neq l$. $\square$

3 if $M = [b_1\ b_2 \cdots b_k]$ then $\left.\begin{matrix} M[x]_B = x \\ M[y]_B = y \end{matrix}\right\} \Rightarrow M([x]_B + [y]_B) = x + y$

but $M[x+y]_B = x+y$

Comment: this argument tells us that if H has a basis with k elements and $v_1, v_2, \ldots, v_\ell \in H$ where $\boxed{\ell > k}$ then these vectors must be linearly dependent

(we already observed this when $H = \mathbb{R}^n$, $k = n$)

of subspace

3 $\boxed{\text{Dimension}}$

Let $\mathcal{B} = (b_1, b_2, \ldots, b_k)$ be an ordered basis of a subspace H of $\mathbb{R}^n$.

The function $H \to \mathbb{R}^k$ with the formula $v \mapsto [v]_{\mathcal{B}}$ is linear and invertible.

Thus H "looks the same as" $\mathbb{R}^k$. For this reason we say that H is *k-dimensional*.

Definition. The *dimension* of a subspace H is the number of vectors in any basis.

We denote the dimension of H by $\dim H$. This number belongs to $\{0, 1, 2, 3, \ldots\}$.

The only way we can have $\boxed{\dim H = 0 \text{ is if } H = \{0\}}$ is the zero subspace.

Example. We have $\dim \mathbb{R}^n = n$ since the standard basis of $\mathbb{R}^n$ has n elements.

each v_i is arbitrary

$$H = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n \right\}$$

If H is the set of all vectors $\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$, then H is a subspace and $\boxed{\dim H = k}$.

one basis: e_1, e_2, e_3, $\vdots$, e_k (in $\mathbb{R}^n$)

Note that $e_1, e_2, \ldots, e_k \in \mathbb{R}^n$ is a basis for H, but H is **not equal** to $\mathbb{R}^k$.

A line in $\mathbb{R}^2$ (or $\mathbb{R}^n$) through the origin is **defined** to be a 1-dimensional subspace.

the fact that H "looks like" $\mathbb{R}^k$

is reflecting the equality $\dim H = \dim \mathbb{R}^k = k$

Subspaces have dimensions, matrices have ranks

Let A be an $m \times n$ matrix.

The processes we gave to construct bases of $\operatorname{Nul} A$ and $\operatorname{Col} A$ imply that:

1 **Corollary.** The dimension of $\operatorname{Nul} A$ is the number of free variables in $Ax = 0$.

= # non-pivot columns

2 **Corollary.** The dimension of $\operatorname{Col} A$ is the number of pivot columns in A.

= # of basic variables in $Ax = 0$

There is a special name for the dimension of the column space of a matrix:

Definition. The *rank* of a matrix A is $\operatorname{rank} A = \dim \operatorname{Col} A$.

not really any widespread term for $\dim \operatorname{Nul} A$

(maybe called "nullity")

Putting everything together gives the following pair of important results.

Theorem (*Rank-nullity theorem*)**.** If A is a matrix with n columns then

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

Proof. Number of free variables in $Ax = 0$ is the number non-pivot columns in A. Therefore rank A + dim Nul A is the total number of columns in A. $\square$

Theorem (*Basis theorem*)**.** If H is a subspace of $\mathbb{R}^n$ with $\dim H = p$ then

1. Any set of p linearly independent vectors in H is a basis for H.
2. Any set of p vectors in H whose span is H is a basis for H.

For proof see lecture notes

Corollary. If H is an n-dimensional subspace of $\mathbb{R}^n$ then $H = \mathbb{R}^n$.

any spanning set of size dim H is a basis

also, any linearly independent set of size dim H is a basis

idea: dim is correct way to measure size of a subspace

$\subset$ (same) $\subseteq$

set containment symbols

Recall that if U and V are two sets then we write "$U \subset V$" or " $U \subseteq V$" to mean that every element of U is also an element of V.

Both notations mean the same thing. If $U \subseteq V$ then it could be true that $U = V$.

On the other hand, writing "$U \subsetneq V$" means "$U \subseteq V$ but $U \neq V$." $\subsetneq$

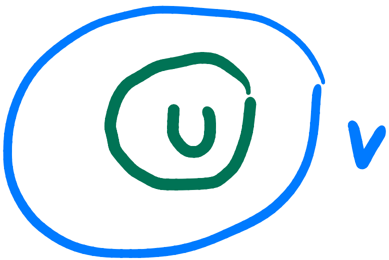
It holds that $U = V$ if and only if we have both $U \subseteq V$ and $V \subseteq U$.

Corollary. If $U, V \subseteq \mathbb{R}^n$ are subspaces with $U \subsetneq V$ then

$$\dim U < \dim V \leq n.$$

Proof. If $j = \dim V \leq \dim U = k$ and $u_1, u_2, \ldots, u_k$ is a basis for U, then $u_1, u_2, \ldots, u_j$ would be linearly independent and therefore a basis for V.

But then $V \subseteq U$ which would imply $U = V$ if also $U \subseteq V$. $\square$



$\Rightarrow$ $\dim U < \dim V$

same data

invertible matrices $\longleftrightarrow$ ordered bases of $\mathbb{R}^n$

as

square

Corollary. Let A be an $n \times n$ matrix. The following are equivalent:

(a) A is invertible.

(b) The columns of A form a basis for $\mathbb{R}^n$.

(c) $\operatorname{rank} A = \dim \operatorname{Col} A = n$. $\Rightarrow$ columns span $\mathbb{R}^n$ $\Rightarrow$ hence a basis for $\mathbb{R}^n$

(d) $\dim \operatorname{Nul} A = 0$. $\Rightarrow$ columns are independent $\Rightarrow$ hence a basis

Proof. We have already seen that (a) and (b) are equivalent.

(c) holds if and only if the columns of A span $\mathbb{R}^n$ which is equivalent to (a).

(d) holds iff columns of A are linearly independent which is equivalent to (a). $\square$