MATH 2121 - Lecture #14

* Note this week's offline + online HW

Outline:

1. Review + more on abstract vector spaces
2. Eigenvalues + eigenvectors

"concrete" vector space $\leftrightarrow$ $\mathbb{R}^n$ and its subspaces

(abstract)

1 Last time: vector spaces

A *(real)* $\boxed{\textit{vector space } V}$ is a set containing a *zero vector*, denoted 0, with *vector addition* and *scalar multiplication* operations that let us produce new vectors $\boxed{u+v} \in V$ and $\boxed{cv} \in V$ from given elements $u, v \in V$ and $c \in \mathbb{R}$.

Several conditions must be satisfied so that these operations behave exactly like vector addition and scalar multiplication for $\mathbb{R}^n$. Most importantly, we require that

1. $u + v = v + u$ and $(u + v) + w = u + (v + w)$.
2. $v - v = 0$ where we define $u - v = u + (-1)v$.
3. $v + 0 = v$
4. $cv = v$ if $c = 1$.

There are a few other more conditions to give the full definition (see the notes).

By convention, we refer to elements of vector spaces as *vectors*.

motivating example

Example. All subspace of $\mathbb{R}^n$ are vector spaces, with the usual zero vector and vector operations.

Example. The set of $m \times n$ matrices is a vector space, with the usual addition and scalar multiplication operations.

The zero vector in this vector space is the $m \times n$ zero matrix.

easy new example

source of many constructions of vector spaces

Common vector spaces are subspaces of $\mathbb{R}^n$ or subspaces of the following:

Proposition. Let X be a set and let V be a vector space. (e.g., $V = \mathbb{R}$)

Then the set $\mathsf{Functions}(X, V)$ of all functions $f : X \to V$ is a vector space with

$$\left.\begin{aligned} f + g &= (\text{ the function that maps } x \mapsto f(x) + g(x) \text{ for } x \in X\ ), \\ cf &= (\text{ the function that maps } x \mapsto c \cdot f(x) \text{ for } x \in X\ ), \\ 0 &= (\text{ the function that maps } x \mapsto 0 \in V \text{ for } x \in X\ ), \end{aligned}\right\} \text{pointwise operations}$$

for $f, g \in \mathsf{Functions}(X, V)$ and $c \in \mathbb{R}$.

useful to think through why

$\mathbb{R}^n$ "=" $\text{Functions}(\{1,2,3,\dots,n\}, \mathbb{R})$

$$\begin{bmatrix} f(1) \\ \vdots \\ f(n) \end{bmatrix} \longleftrightarrow f$$

lots of vocab, mostly repeating concepts related to subspaces of $\mathbb{R}^n$

Definition. The definitions of a *subspace* of a vector space and of *linear transformations* between vector spaces are just like the ones we saw for subspaces of $\mathbb{R}^n$:

- A subset $H \subseteq V$ is a *subspace* if (V is any vector space)

  $$0 \in H \quad \text{and} \quad u + v \in H \text{ for all } u, v \in H \quad \text{and} \quad cv \in H \text{ for all } c \in \mathbb{R}.$$

  a subspace H is itself a vector space

- A function $f : U \to V$ is *linear* if (U, V both vector space)

  $$f(\underbrace{u + v}_{\text{happening in } U}) = \underbrace{f(u) + f(v)}_{\text{happening in } V} \text{ for all } u, v \in U \quad \text{and} \quad f(cv) = cf(v) \text{ for all } c \in \mathbb{R}.$$

Proposition. If U, V, W are vector spaces and $f : V \to W$ and $g : U \to V$ are linear then $f \circ g : U \to W$ is also linear, where $f \circ g(x) = f(g(x))$ for $x \in U$.

$e_1, e_2 \ldots e_n$

Example. If U and V are vector spaces then let $\mathsf{Lin}(U, V)$ be the set of linear functions $f : U \to V$. Then $\mathsf{Lin}(U, V)$ is a subspace of $\mathsf{Functions}(U, V)$.

key difference between linear functions $\mathbb{R}^n \to \mathbb{R}^m$ versus $U \to V$ is that we no longer have a notion of standard matrix (as there is no standard basis) vectors of form

$c_1 v_1 + c_2 v_2 + \dots + c_k v_k$

where $c_1, c_2, \dots, c_k \in \mathbb{R}$, $v_1, \dots, v_k \in V$

Let V be a vector space. The definitions of *linear combinations*, *span* and *linear independence* for vectors in V are the same as for vectors in $\mathbb{R}^n$.

Remember that we can only evaluate the linear combination $c_1 v_1 + c_2 v_2 + \dots$ if it is a finite sum, or if there are finitely many nonzero scalars $c_i \neq 0$.

Example. The subspace of polynomials in $\mathsf{Functions}(\mathbb{R}, \mathbb{R})$ is the $\boxed{\text{span of } 1, x, x^2, x^3, \dots}$.

But $e^x = 1 + x + \frac{1}{2}x + \frac{1}{6}x^2 + \frac{1}{24}x^3 + \cdots + \frac{1}{n!}x^n + \dots$ is not in this subspace.

gives polynomials

refers to vectors $v_1, \dots, v_k \in V$
which only produce
zero vector as a linear comb.

$0 = c_1 v_1 + c_2 v_2 + \dots + c_k v_k$

if we have $c_1 = c_2 = \dots = c_k = 0$

Definition. A *basis* of a vector space V is a subset of linearly independent vectors whose span is V.

Saying $b_1, b_2, b_3, \dots$ is a basis for V is the same as saying that for each $v \in V$, there a unique coefficients $x_1, x_2, x_3, \dots \in \mathbb{R}$, **all but finitely many of which are zero**, such that $v = x_1 b_1 + x_2 b_2 + x_3 b_3 + \dots$.

(can ignore if basis is finite)

Theorem. Let V be a vector space. Then V has at least one basis, and every basis of V has the same number of elements (but this could be infinite).

Definition. The *dimension* of a vector space V is the number $\dim V$ of elements in any of its bases.

Example. If X is a finite set then $\dim \mathsf{Functions}(X, \mathbb{R}) = |X|$.

Ex $\mathbb{R}^{m \times n}$ = vector space of $m \times n$ matrices

$\dim(\mathbb{R}^{m\times n}) = mn$ as a basis is $\left\{ E_{ij} \;\middle|\; \begin{matrix} 1 \le i \le m \\ 1 \le j \le n \end{matrix} \right\}$

$E_{ij} = [\cdots 1 \cdots]$ single nonzero entry (i,j) equal to 1

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{E_{11}} + b\underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{E_{12}} + c\underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{E_{21}} + d\underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{E_{22}}$$

So it's clear that $\mathbb{R}^{2\times 2} = \mathbb{R}\text{-span}[E_{11}, E_{12}, E_{21}, E_{22}]$

also these matrices are linearly independent

So only way to make the LHS $= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is by

taking $a = b = c = d = 0$

Why is $\dim(\text{Functions}(X, \mathbb{R})) = |X|$ (size of X)?

Here is a basis for

For each $y \in X$ define $b_y : X \to \mathbb{R}$ by formula

$$b_y(x) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

Claim: $\{f_y \mid y \in X\}$ is a basis for

spanning: $f \underset{\text{check}}{=} \sum_{y \in X} \underset{\in \mathbb{R}}{f(y)} \, \underset{\text{a function}}{b_y}$

(for more details, see lecture notes)

think of a function as a collection of mail slots

each labeled by "address" $x \in X$

each containing "mail" $f(x) \in Y$

$b_y \leftrightarrow$ all slots empty except slot y, which has 1 in it

$\dim(V) = n < \infty$

vector space of linear functions $V \to \mathbb{R}$

2 More on dimension

If V is a finite-dimensional vector space then $\boxed{\dim \mathsf{Lin}(V, \mathbb{R}) = \dim V}$.

To see this, suppose $b_1, b_2, \dots, b_n$ is a basis for V.

dual basis

A basis for $\mathsf{Lin}(V, \mathbb{R})$ is given by the linear functions $\phi_1, \phi_2, \dots, \phi_n : V \to \mathbb{R}$ with

$$\phi_i(x_1 b_1 + x_2 b_2 + \dots x_n b_n) = x_i \qquad \text{for } x_1, x_2, \dots, x_n \in \mathbb{R}.$$

The unique way to express any linear $f : V \to \mathbb{R}$ as a combination of these is

$$f = f(b_1)\phi_1 + f(b_2)\phi_2 + \cdots + f(b_n)\phi_n.$$

$e_i = \begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix}$, $e_i^T = [\cdots 1 \cdots]$

When $V = \mathbb{R}^n$, we can think of $\mathsf{Lin}(\mathbb{R}^n, \mathbb{R})$ as the vector space of $1 \times n$ matrices.

If $b_1 = e_1$, $b_2 = e_2$, $\dots$, $b_n = e_n$ is the standard basis, then $\phi_i = e_i^\top$.

We know $\mathrm{Lin}(\mathbb{R}^n, \mathbb{R}) = \mathbb{R}^{1 \times n}$ ($1 \times n$ matrices)

from prev example, this has basis

$\{E_{11}, E_{12}, \dots, E_{12}\} = \{e_1^T, e_2^T, \dots, e_n^T\}$

Vector space generalizations of Col(A) and Nul(A)

Definition. Suppose U and V are vector spaces and $\boxed{f : U \to V \text{ is a linear function.}}$ Define $\boxed{\mathsf{range}(f) = \{f(x) : x \in U\} \subseteq V}$ and $\boxed{\mathsf{kernel}(f) = \{x \in U : f(x) = 0\} \subseteq U}$.

These subspaces generalize the column space and null space of a matrix.

We have a version of the rank-nullity theorem for arbitrary vector spaces:

Theorem (Rank-Nullity Theorem)**.** If $\dim U < \infty$ then

$$\dim \mathsf{range}(f) + \dim \mathsf{kernel}(f) = \dim U.$$

This specializes to our earlier statement about matrices when $U = \mathbb{R}^n$ and $V = \mathbb{R}^m$.

For a self-contained proof, see the lecture notes.

if $U = \mathbb{R}^n$, $V = \mathbb{R}^m$, A = (standard matrix of f)

then $\dim \text{range}(f) + \dim \text{kernel}(f) = \dim U$

$\dim \text{Col}(A) + \dim \text{Nul}(A) = n$ = # columns of A

Ex We have discussed a basis for $\mathbb{R}^{3\times 3}$. What about these subspaces?

A basis for 3×3 symmetric matrices $(X^T = X)$: $\left\{ \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} : a,b,c,d,e,f \in \mathbb{R} \right\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

A basis for 3×3 skew-symmetric matrices $(X^T = -X)$: $\left\{ \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix} \right\}$

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

only defined for square matrices

3 Eigenvectors and eigenvalues

Let A be a square $n \times n$ matrix.

Definition. An *eigenvector* of A is a **nonzero** vector $v \in \mathbb{R}^n$ such that

$$\boxed{Av = \lambda v}$$

for a scalar $\lambda \in \mathbb{R}$

for a number $\lambda \in \mathbb{R}$. (λ is the Greek letter "lambda.")

The number λ is called the *eigenvalue* of A for the eigenvector v.

must have $v \neq 0$

Why require nonzero? Because if $v = 0$ then $Av = \lambda v = 0$ for all numbers $\lambda \in \mathbb{R}$.

The number 0 is allowed to be an eigenvalue of A, however.

can have $\lambda = 0$

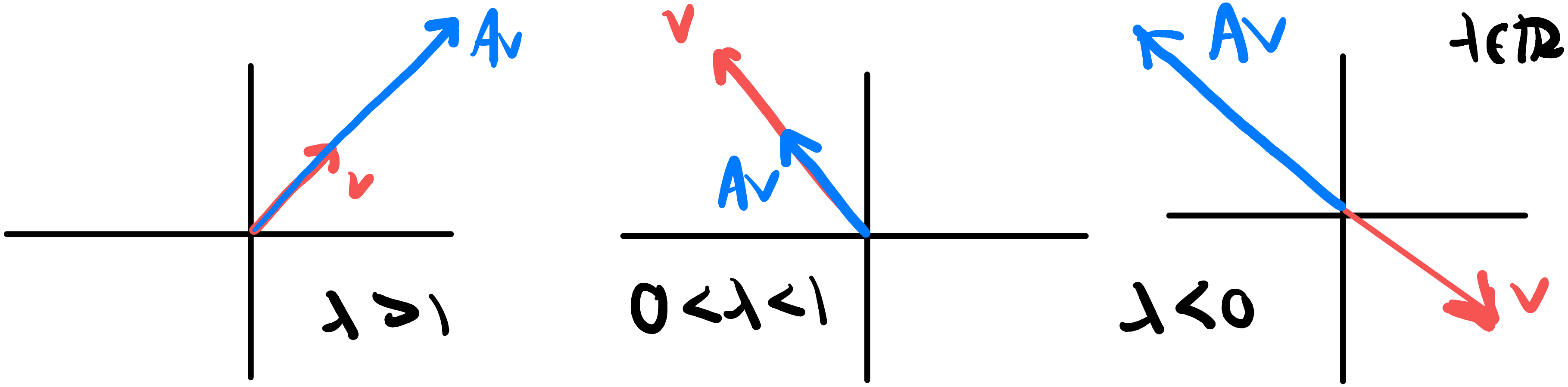
"eigen" is German for "same"

v is an eigenvector for A if v and Av are on same line

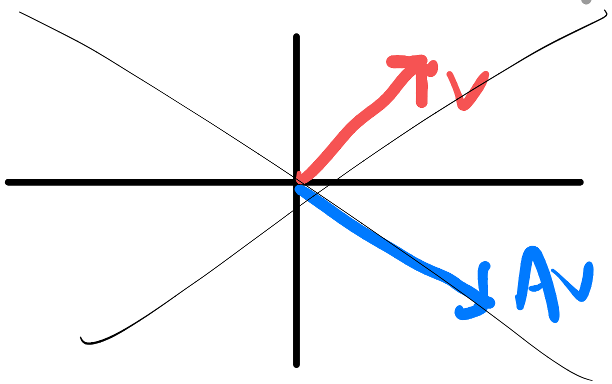
Suppose A is 2×2 matrix with eigenvector $v \in \mathbb{R}^2$

so $Av = \lambda v$ for some $\lambda \in \mathbb{R}$

The vectors Av and v might look like this:



but this picture is NOT possible:



(vectors v and Av are NOT on SAME line)

Ex What are eigenvectors of A with eigenvalue $\lambda = 0$?

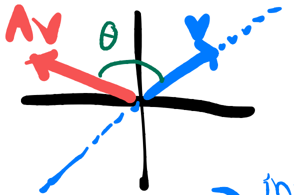
these are the nonzero vectors v with $Av = 0v = 0$

in other words: the $\boxed{\text{nonzero elements of Nul } A}$

Ex What are the eigenvectors of a rotation matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$?

recall that $Av = $ (v rotated ccw by θ radians)

so if θ is not a multiple of π then Av is NOT on same line as v:



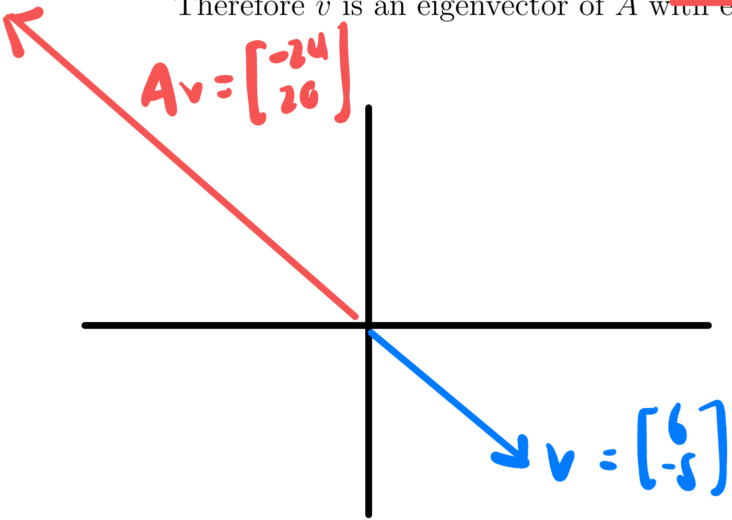
$\Rightarrow$ $\boxed{\text{if } \theta \notin [\ldots, -\pi, 0, \pi, 2\pi, 3\pi, \ldots] \text{ then } A \text{ has no eigenvectors}}$

But if $\theta = 0$ then $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and if $\theta = \pi$ then $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ } in these cases all nonzero vectors are eigenvectors

Example. Given A and v, can check whether v is an eigenvector by computing Av.

If $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ and $v = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$ then $\boxed{Av = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4v.}$

Therefore v is an eigenvector of A with eigenvalue -4.



hence v is an eigenvector of A with eigenvalue $\lambda = -4$

Example. What are the eigenvectors of the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$?

If $v \in \mathbb{R}^4$ were an eigenvector with eigenvalue λ then

$$Av = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_3 \\ v_4 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}.$$

only holds if

$0 = \lambda v_4$

$v_4 = \lambda v_3$

$v_3 = \lambda v_2$

$v_2 = \lambda,$

The last equation implies that $0 = \lambda v_4$ and $v_4 = \lambda v_3$ and $v_3 = \lambda v_2$ and $v_2 = \lambda v_1$. So

$$\boxed{0 = \lambda v_4 = \lambda^2 v_3 = \lambda^3 v_2 = \lambda^4 v_1.}$$

If $\lambda \neq 0$ then this would mean that $v_1 = v_2 = v_3 = v_4 = 0$, but remember that v should be nonzero. Therefore the only possible eigenvalue of A is $\boxed{\lambda = 0}$.

then to get an eigenvector: set $v_1 = 1,\ v_2 = v_3 = v_4 = 0$

The eigenvectors of $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ with eigenvalue 0 are

$$\boxed{v = \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{where } x \neq 0.}$$

Check: $A \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} x \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

"λ is an eigenvalue of A" means: "there is some $0 \neq v \in \mathbb{R}^n$ such that $Av = \lambda v$."

Recall that I_n denotes the $n \times n$ identity matrix. We abbreviate by setting $I = I_n$. $= \begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}$

Proposition. $\lambda \in \mathbb{R}$ is an eigenvalue of A if and only if $A - \lambda I$ is not invertible.

Proof. $Ax = \lambda x$ has a nonzero solution $x \in \mathbb{R}^n$ if and only if $(A - \lambda I)x = 0$ has a nonzero solution, which occurs if and only if $A - \lambda I$ is not invertible. $\square$

$(A - \lambda I)x = Ax - \lambda I x = Ax - \lambda x$

Example. If $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ then

$\lambda = 7$

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \mathsf{RREF}(A - 7I).$$

As $\mathsf{RREF}(A - 7I) \neq I$, the matrix $A - 7I$ is not invertible so 7 is an eigenvalue of A.

in fact, the eigenvectors for A with eigenvalue λ are just the nonzero elements of $\mathrm{Nul}(A - \lambda I)$

$\boxed{\textbf{Corollary.} \text{ A number } \lambda \in \mathbb{R} \text{ is an eigenvalue of } A \text{ if and only if } \det(A - \lambda I) = 0.}$

Proof. Remember that $A - \lambda I$ is not invertible if and only if $\det(A - \lambda I) = 0$. $\square$

Another way of defining an eigenvector: the eigenvectors of A with eigenvalue λ are precisely the nonzero elements of the null space $\mathrm{Nul}(A - \lambda I)$.

Since we know how to construct a basis for the null space of any matrix, we also know how to find all eigenvectors of a matrix for any given eigenvalue.

$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$ $\lambda = 7$

Example. In the previous example, $\boxed{\mathsf{RREF}(A - 7I) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}}$ so $Ax = 7x$ if and only if $(A - 7I)x = 0$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ where $x_1 - x_2 = 0$. In this linear system, x_2 is a free variable, and we can rewrite x as $x = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

This means $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a basis for $\mathrm{Nul}(A - 7I)$.

So every eigenvector of A with eigenvalue 7 has the form $\begin{bmatrix} a \\ a \end{bmatrix}$ for some $a \in \mathbb{R}$. $(a \neq 0)$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for A w/ eigenvalue 7

Check: $\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ✓

$$\text{Nul}(A - \lambda I) = \{x \in \mathbb{R}^n \mid Ax = \lambda x\}$$

One calls the set of all $v \in \mathbb{R}^n$ with $Av = \lambda v$ the *eigenspace* of A for λ. We also call this the $\boxed{\lambda\textit{-eigenspace}}$ of A. This is just the null space of $A - \lambda I$.

A number is an eigenvalue of A if and only if the corresponding eigenspace is nonzero (that is, contains a nonzero vector).

λ is an eigenvalue for A iff λ-eigenspace is NONZERO

(contains more than just the zero vector)

the λ-eigenspace of A is defined when λ is not an eigenvalue, but then it's the zero subspace

Upgrade to previous 2x2 example: (same techniques)

Example. Suppose we were told that $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ has 2 as an eigenvalue.

$\lambda = 2$

To find a basis for the 2-eigenspace of A, we row reduce

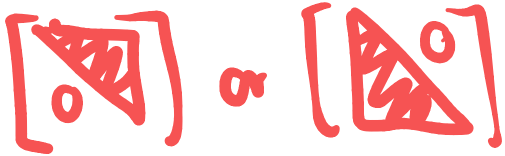
$$A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(A - 2I).$$

Thus $Ax = 2x$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ where $x_1 - \frac{1}{2}x_2 + 3x_3 = 0$, meaning

$$x = \begin{bmatrix} \frac{1}{2}x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow \boxed{\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \text{ is basis for 2-eigenspace.}}$$

all eigenvectors for A with eigval $\lambda = 2$ have form $a \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

with $a \neq 0$ or $b \neq 0$



Recall that a matrix is *triangular* if its nonzero entries all appear on or above the main diagonal, or all appear on or below the main diagonal.

Theorem. The eigenvalues of a triangular square matrix A are its diagonal entries.

Proof. If A has diagonal entries $d_1, d_2, \ldots, d_n$ then $A - \lambda I$ is triangular with diagonal entries $d_1 - \lambda$, $d_2 - \lambda$, $\ldots$, $d_n - \lambda$, so

$$\det(A - \lambda I) = (d_1 - \lambda)(d_2 - \lambda) \cdots (d_n - \lambda)$$

which is zero if and only if $\lambda \in \{d_1, d_2, \ldots, d_n\}$. $\square$

Example. The eigenvalues of the matrix $\begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ are 3, 0, and 2.