MATH 2121 - Lecture 15

Outline:

- Eigenvectors and eigenvalues
- intro to "eigenvalue decomposition" of a matrix / "diagonalization"

$I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$, $Iv = v$ for all $v \in \mathbb{R}^n$

1 Eigenvector and eigenvalues

square

Everywhere is this lecture, n is a positive integer and A is an $n \times n$ matrix. Let I denote the $n \times n$ identity matrix. Let λ be a number.

Definition. A vector $v \in \mathbb{R}^n$ is an *eigenvector* for A with *eigenvalue* λ if

$$v \neq 0 \quad \text{and} \quad Av = \lambda v.$$

equiv: v is a nonzero elem of $\mathrm{Nul}(A - \lambda I)$

The set of all $v \in \mathbb{R}^n$ with $Av = \lambda v$ is the *λ-eigenspace* of A for λ.

Note that this is equal to the nullspace of $A - \lambda I$.

a nonzero vector that, when multiplied by A, gives a scalar multiple of itself

Warmup examples:

If $A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$ is diagonal, then

all elementary basis vectors $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$,

$e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

are eigenvectors: notice $Ae_i = a_i e_i$

(this property holds for any diagonal matrix)

If $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ then

$$A \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} v_2 \\ v_3 \\ v_4 \\ v_1 \end{bmatrix}$$

So one eigenvector is $v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(this has eigenvalue $\lambda = 1$ as $Av = v$)

Fact to remember: a square matrix A is invertible if and only if $\mathrm{Nul}(A) = 0$ iff $\det A \neq 0$

Proposition. Let λ be a number. The following are equivalent:

1. $\boxed{\text{There exists an eigenvector } v \in \mathbb{R}^n \text{ for } A \text{ with eigenvalue } \lambda.}$ Say "λ is an eigenvalue of A"

   (Remember that eigenvectors must be nonzero.)

2. The matrix $A - \lambda I$ is not invertible.
3. $\det(A - \lambda I) = 0$.
4. The λ-eigenspace for A contains a nonzero vector. $\leftrightarrow$ $\mathrm{Nul}(A-\lambda I) \neq 0$

As usual, a matrix is $\boxed{\textit{triangular}}$ if it is upper-triangular or lower-triangular.

The *characteristic polynomial* of a square matrix A is $\boxed{\det(A - xI)}$.

Theorem. The eigenvalues of a triangular square matrix A are its $\boxed{\text{diagonal entries}}$. If these are $d_1, \ldots, d_n$ then characteristic polynomial of A is $(d_1 - x) \cdots (d_n - x)$.

by the prop, λ is an eigenvalue for A precisely when $x = \lambda$ is a solution to $\det(A - xI) = 0$

Ex Suppose $A = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$ (this is triangular)

its eigenvalues are 1 and 2

what's an eigenvector for eigenvalue $\lambda = 1$?

$$\boxed{v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

as $\begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

what's an eigenvector for eigenvalue $\lambda = 2$?

this will be a nonzero elem of null space of

$$A - 2I = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 0 & 0 \end{bmatrix}$$

one example: $\boxed{v = \begin{bmatrix} 5 \\ 1 \end{bmatrix}}$

A new result and useful principle:

The following is true for all square matrices, not just triangular ones.

Theorem. Suppose $\lambda_1, \lambda_2, \ldots, \lambda_r$ are **distinct** eigenvalues for A.
Let $v_1, v_2, \ldots, v_r \in \mathbb{R}^n$ be the corresponding eigenvectors, so that $\boxed{Av_i = \lambda_i v_i}$
Then the vectors $v_1, v_2, \ldots v_r$ are linearly independent.

"distinct" means that $\lambda_1 \neq \lambda_2$, $\lambda_1 \neq \lambda_3$, ... $\lambda_1 \neq \lambda_r$

$\lambda_2 \neq \lambda_3$, $\lambda_2 \neq \lambda_4$, ..., $\lambda_2 \neq \lambda_r$

etc

(so if $i \neq j$ then $\lambda_i \neq \lambda_j$)

The detailed proof is in the lecture notes.

Here is an intuitive reason to believe that eigenvectors with distinct eigenvalues are linearly independent.

Suppose $\lambda_1 >$ (all other eigenvalues) but we can express

$$0 = c_1 v_1 + c_2 v_2 + \dots + c_r v_r \overset{\text{def}}{=} w$$

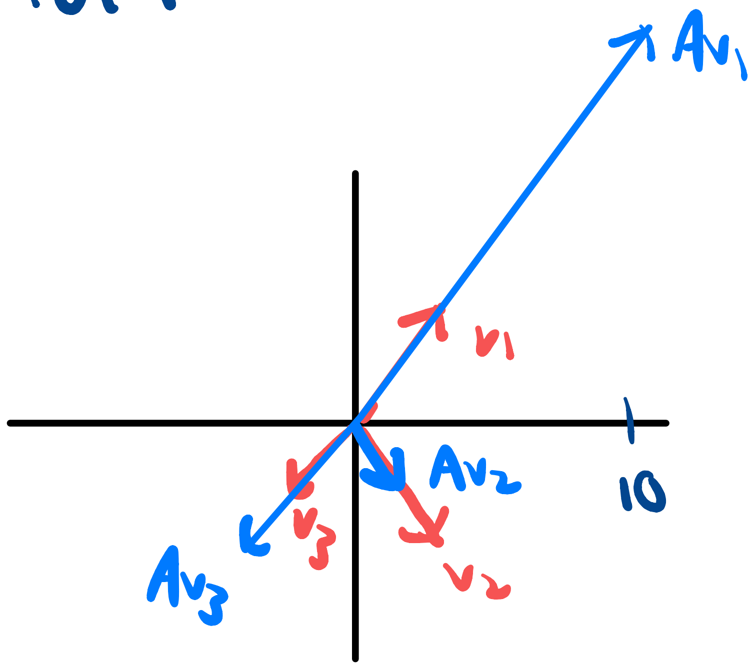
Then $A^{10000} w = c_1 \lambda_1^{10000} v_1 + c_2 \lambda_2^{10000} v_2 + \dots + c_r \lambda_r^{10000} v_r \approx c_1 \lambda_1^{10000} v_1$

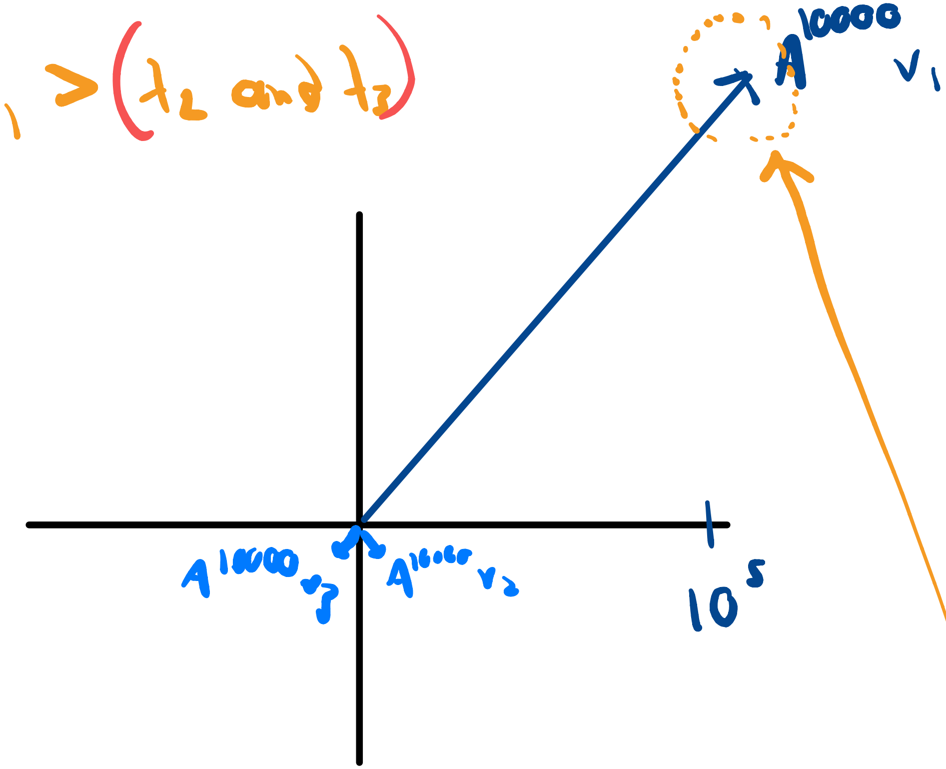
but also $A^{10000} w = A^{10000} 0 = 0$ so must have $c_1 = 0$

(then continue argument)

Picture:

$\lambda_1 > (\lambda_2 \text{ and } \lambda_3)$





if $c_1v_1 + c_2v_2 + c_3v_3 = 0$

then $\underbrace{c_1\lambda_1^{10000}v_1}_{\text{largest term}} + c_2\lambda_2^{10000}v_2 + c_3\lambda_3^{10000}v_3$ will be

Let x be a variable. The eigenvalues of A are precisely the solutions to the equation $\boxed{\det(A - xI) = 0}$ which we call the *characteristic equation* for A.

To find eigenvalues: try to factor $\det(A-xI)$

Example. The matrix

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

has characteristic polynomial $\det(A - xI) = (5 - x)^2(3 - x)(1 - x)$.

Since $(5 - x)^2$ divides $\det(A - xI)$ but $(5 - x)^3$ does not divide $\det(A - xI)$, we say that 5 is an eigenvalue of A with *algebraic multiplicity* 2.

The other eigenvalues 1 and 3 have *algebraic multiplicity* 1.

The $\boxed{\textit{algebraic multiplicity}}$ of an eigenvalue λ for A is the unique integer $m \geq 1$ such that $(\lambda - x)^m$ divides $\det(A - xI)$ but $(\lambda - x)^{m+1}$ does not divide $\det(A - xI)$.

a more useful geometric multiplicity of λ is $\dim(A - \lambda I)$ which turns out to be $\leq$ (algebr. multiplicity)

We consider the following example in more depth.

Example. Consider the matrix

$$A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

A is triangular, so $\boxed{\det(A - xI) = (1-x)(2-x)(3-x)}$ and A's eigenvalues are $\boxed{1, 2, 3.}$

Each eigenvalue in this example has algebraic multiplicity 1.

We compute the corresponding eigenspaces next.

1-eigenspace. Eigenvectors of A w/eigenvalue 1 are nonzero elements of Nul$(A-I)$.

$$A-I=\begin{bmatrix}0&5&4\\0&1&0\\0&0&2\end{bmatrix}\sim\begin{bmatrix}0&1&0\\&5&4\\&&2\end{bmatrix}\sim\begin{bmatrix}0&1&0\\&0&4\\&&2\end{bmatrix}\sim\begin{bmatrix}0&1&0\\&0&1\\&&0\end{bmatrix}=\mathsf{RREF}(A-I)\Leftrightarrow\begin{cases}x_2=0\\x_3=0\\0=0\end{cases}$$

This shows that $x\in\text{Nul}(A-I)$ if and only if $x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\0\\0\end{bmatrix}=x_1\begin{bmatrix}1\\0\\0\end{bmatrix}$.

So $\begin{bmatrix}1\\0\\0\end{bmatrix}$ is a basis for Nul$(A-I)$.

$$\boxed{\text{Thus all eigenvectors of } A \text{ with } \underline{\text{eigenvalue 1}} \text{ are nonzero scalar multiples of } \begin{bmatrix}1\\0\\0\end{bmatrix}.}$$

Check: $\begin{bmatrix}1&5&4\\0&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}$ ✓

2-eigenspace. Eigenvectors of A w/eigenval. 2 are nonzero elements of $\mathrm{Nul}(A-2I)$.

$$A-2I = \begin{bmatrix} -1 & 5 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(A-2I). \Leftrightarrow \begin{cases} x_1 - 5x_2 = 0 \\ x_3 = 0 \\ 0 = 0 \end{cases}$$

This shows that $\underline{x \in \mathrm{Nul}(A-2I)}$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$.

So $\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ is a basis for $\mathrm{Nul}(A-2I)$.

$$\boxed{\text{All eigenvectors of } A \text{ with eigenvalue 2 are nonzero scalar multiples of } \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}.}$$

Check: $\begin{bmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5+5+0 \\ 0+2+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$

3-eigenspace. Eigenvectors of A w/eigenval. 3 are nonzero elements of $\mathrm{Nul}(A-3I)$.

$$A - 3I = \begin{bmatrix} -2 & 5 & 4 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & 4 \\ & 1 & 0 \\ & & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ & 1 & 0 \\ & & 0 \end{bmatrix} = \mathsf{RREF}(A-3I). \Leftrightarrow \begin{cases} x_1 - 2x_3 = 0 \\ x_2 = 0 \\ 0 = 0 \end{cases}$$

This shows that $x \in \mathrm{Nul}(A-3I)$ if and only if $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

So $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ is a basis for $\mathrm{Nul}(A-3I)$.

All eigenvectors of A with eigenvalue 3 are nonzero scalar multiples of $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

✓

Check: $\begin{bmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+0+4 \\ 0+0+0 \\ 0+0+3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

Since the eigenvalues 1, 2, 3, are distinct, the eigenvectors $\overset{u}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \overset{v}{\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}}, \overset{w}{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}$ are linearly independent.

Consider the **invertible** and **diagonal** matrices

$$P = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \mathrm{diag}(1,2,3)$$

It turns out that $\boxed{A = PDP^{-1}}$. How can we check this without multiplication?

we define $P = [u \ v \ w]$

P is invertible because its columns are lin. independent

(and the matrix is square)

Recall: $Pe_i = (i^{\text{th}}$ column of $P)$ $e_1 = P^{-1}(Pe_1) = P^{-1}(u) = P^{-1}u$

Similarly, $e_2 = P^{-1}v$, $e_3 = P^{-1}w$

Let $u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$, and $w = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$ be the columns of $P = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Also let $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

*** Then $\boxed{Pe_1 = u,\ Pe_2 = v, \text{ and } Pe_3 = w} \rightarrow \boxed{P^{-1}u = e_1,\ P^{-1}v = e_2, \text{ and } P^{-1}w = e_3}$.

But for $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ we have $De_1 = e_1$ and $De_2 = 2e_2$ and $De_3 = 3e_3$.

So $\underline{PDP^{-1}u = u}$ and $\underline{PDP^{-1}v = 2v}$ and $\underline{PDP^{-1}w = 3w}$.

At the same time $\underline{Au = u}$ and $\underline{Av = 2v}$ and $\underline{Aw = 3w}$.

Since u, v, w **are a basis for** $\mathbb{R}^3$, we must have $\boxed{A = PDP^{-1}}$.

checking that PDP^{-1} and A multiplied with u, v, w give same result.

Remark If A is any $n \times n$ matrix and we can find an invertible matrix $P = [v_1 \; v_2 \cdots v_n]$ $(v_i \in \mathbb{R}^n)$ and a diagonal matrix $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

such that $\boxed{A = PDP^{-1}}$ ← call this an eigenvector decomposition of A

then ① $\lambda_1, \lambda_2 \cdots \lambda_n$ are the eigenvalues of A

② each v_i is an eigenvector for A because

$$Av_i = PD\underbrace{P^{-1}v_i}_{= e_i \text{ as } Pe_i = v_i} = PDe_i = P\lambda_i e_i = \lambda_i P e_i = \lambda_i v_i$$

If you can compute $A = PDP^{-1}$ (D diagonal)

then

One application of this decomposition: (a simple formula for powers A^n of A.)

Define $A^0 = I$, $A^1 = A$, $A^2 = AA$, $A^3 = AAA$, and so on.

Lemma. For any integer $n \geq 0$ we have $\boxed{A^n = (PDP^{-1})^n = PD^nP^{-1}.}$

Proof. Do some small examples and convince yourself that the pattern continues:

$$A^2 = AA = PDP^{-1}PDP^{-1} = PDIDP^{-1} = PD^2P^{-1}$$

$$A^3 = A^2A = PD^2P^{-1}PDP^{-1} = PD^2IDP^{-1} = PD^3P^{-1}$$

$$A^4 = A^3A = PD^3P^{-1}PDP^{-1} = PD^3IDP^{-1} = PD^4P^{-1}$$

$$\vdots$$

and so on. ∎

Also:

$$\begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}^N = \begin{bmatrix} d_1^N & & & 0 \\ & d_2^N & & \\ & & \ddots & \\ 0 & & & d_n^N \end{bmatrix}$$

for any $N > 0$

Lemma. For any integer $n \geq 0$ we have

$$D^n = \begin{bmatrix} 1^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix}.$$

Proof. To multiply diagonal matrices we just multiply the entries in the corresponding diagonal positions:

$$\begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_k \end{bmatrix} \begin{bmatrix} y_1 & & & \\ & y_2 & & \\ & & \ddots & \\ & & & y_k \end{bmatrix} = \begin{bmatrix} x_1y_1 & & & \\ & x_2y_2 & & \\ & & \ddots & \\ & & & x_ky_k \end{bmatrix}.$$

So to evaluate $D^n = DD \cdots D$, just raise each diagonal entry to the nth power. $\square$

Recall: $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = PDP^{-1}$ $\quad D = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}$

By usual algorithm can compute $P^{-1} = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -5 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Putting everything together gives the identity

$$\begin{aligned} A^n = PD^nP^{-1} &= \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & -5 & -2 \\ & 1 & 0 \\ & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 5 \cdot 2^n & 2 \cdot 3^n \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & -5 & -2 \\ & 1 & 0 \\ & & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 5(2^n - 1) & 2(3^n - 1) \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \end{aligned}$$

← completely explicit formula for A^n

for any integer $n>0$

2 Similar matrices

$\underbrace{X = PYP^{-1}}$

Definition. Two $n \times n$ matrices X and Y are $\boxed{\textit{similar}}$ if there exists an invertible $n \times n$ matrix P with $X = PYP^{-1}$.

In this case it also holds that $Y = P^{-1}PYP^{-1}P = P^{-1}XP$.

If X and Y are similar, then we say "X is *similar to* Y" and "Y is *similar to* X."

(find P with $A = PDP^{-1}$)

Above we showed that $A = \begin{bmatrix} 1 & 5 & 4 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ are similar matrices.

There is a special name for this kind of similarity: *diagonalizability*.

Definition. A square matrix X is $\boxed{\textit{diagonalizable}}$ if X is <u>similar to a diagonal matrix</u>

Proposition. An $n \times n$ matrix A is always similar to itself.

(This means that similarity is a *reflexive* relation on square matrices.)

Proof. Since $I = I^{-1}$ we have $A = PAP^{-1}$ for $\boxed{P = I.}$ $\square$

Proposition. Suppose A, B, C are $n \times n$ matrices.

Assume A and B are similar. Assume B and C are also similar.

Then A and C are similar.

(This means that similarity is a *transitive* relation on square matrices.)

Proof. If $\underline{A = PBP^{-1}}$ and $\underline{B = QCQ^{-1}}$ then $A = RCR^{-1}$ for $R = PQ$. $\square$

$$A = (PQ)C(PQ)^{-1} = PQCQ^{-1}P^{-1}$$
$$= PBP^{-1}$$

Theorem. If A and B are similar $n \times n$ matrices then A and B have the same characteristic polynomial and so they have the same eigenvalues.

(Similar matrices usually have different eigenvectors, however.)

Proof. Recall that $\det(XY) = \det(X)\det(Y)$. Assume $A = PBP^{-1}$. Then

$$A - xI = \underbrace{PBP^{-1}}_{=A} - x\underbrace{PIP^{-1}}_{=I} = \underbrace{P(B - xI)P^{-1}}_{\text{factor}}$$

and

$$\Rightarrow \quad \det(A - xI) = \det(P(B - xI)P^{-1}) = \det(P)\det(B - xI)\det(P^{-1}).$$

But

$$\det(\cancel{P})\det(\cancel{P}^{-1}) = \det(PP^{-1}) = \det(I) = 1,$$

so

$$\boxed{\det(A - xI) = \det(B - xI).}$$

□

for similar matrices A and B

Notice, however, that if $A = PBP^{-1}$

and $Bv = \lambda v$ then $Aw = \lambda w$ for $\boxed{w = Pv}$

<u>Pf</u> $Aw = PBP^{-1}(Pv) = PBIv$

$$= PBv$$

$$= P\lambda v$$

$$= \lambda Pv = \lambda w \quad \square$$

So similar matrices A and B have <u>different</u> eigenvectors, (though they are closely related)