MATH 2121 - Lecture #16

Outline:

- Similarity, diagonalization (review)
- Application: formula for Fibonacci numbers
- Diagonalization (for matrices with repeated eigenvalues)

1 Last time: similar and diagonalizable matrices

Let n be a positive integer. Suppose $\boxed{A \text{ is an } n \times n \text{ matrix}}$, $\boxed{v \in \mathbb{R}^n}$, and $\boxed{\lambda \in \mathbb{R}}$.

v an *eigenvector* for A with *eigenvalue* λ if $0 \neq v \in \mathrm{Nul}(A - \lambda I)$, so $Av = \lambda v$.

λ is an eigenvalue of A if there exists some eigenvector with this eigenvalue.

The nullspace $\mathrm{Nul}(A - \lambda I)$ is called the *λ-eigenspace* of A.

characteristic eq

The eigenvalues of A are the solutions to the polynomial equation $\det(A - xI) = 0$.

Fact. Eigenvectors of A with all distinct eigenvalues are linearly independent.

Two $n \times n$ matrices A and B are *similar* if $\boxed{A = PBP^{-1}}$ for some matrix P.

Example. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$.

similar

$\boxed{\textbf{Similar matrices have the same eigenvalues}}$ but usually different eigenvectors.

However, matrices may have the same eigenvalues but not be similar.

A and B both have only one eigval $\lambda = 2$

Example. The matrices

scalar matrix = a constant times I

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$PAP^{-1} = APP^{-1} = A$

as A is scalar

both have only one eigenvalue given by the number 2.

But they are not similar: as $A = 2I$, for every invertible 2×2 matrix P we have

$$PAP^{-1} = 2PIP^{-1} = 2PP^{-1} = 2I = A \neq B.$$

A matrix is *diagonal* if all of its nonzero entries appear in diagonal positions $(1,1), (2,2), \ldots$

A matrix A is *diagonalizable* if it is similar to a diagonal matrix.

* Every diagonal matrix is diagonalizable.

a matrix is similar to itself

this shows A is similar only to itself

"To diagonalize" a matrix A means to find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$

Ex. From HW: two diagonal matrices are similar if their diagonal entries are rearrangements of each other: $D_1 = \begin{bmatrix} 1 & \\ & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & \\ & 1 \end{bmatrix} = D_2$

You can use this to find an invertible matrix P such that $A = PBP^{-1}$ if you can diagonalize $A = P_1 D_1 P_1^{-1}$ and $B = P_2 D_2 P_2^{-1}$

How to diagonalize, and what does it give us:

A matrix A is diagonalizable when $\boxed{A = PDP^{-1}}$ for some $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$.

→ In this case the numbers $\boxed{\lambda_1, \lambda_2, \ldots, \lambda_n}$ are the eigenvalues of A.

Why? The matrices A and D are similar.

→ Also if $P = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}$ then $\boxed{Av_i = \lambda_i v_i}$ for each $i = 1, 2, \ldots, n$.

$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$

Why? We have $Pe_i = v_i$ so $P^{-1}v_i = P^{-1}Pe_i = Ie_i = e_i$. We also have $De_i = \lambda_i e_i$. This means that $Av_i = PDP^{-1}v_i = PDe_i = P(\lambda_i e_i) = \lambda_i Pe_i = \lambda_i v_i$.

→ The columns of P are a basis for $\mathbb{R}^n$ of eigenvectors of A.

Why? We just saw that the columns of P are eigenvectors. They are a basis because P is (invertible) $\Leftrightarrow$ (columns form a basis)

Note: the eigvals of a diagonal / triangular matrix are its diagonal entries

We can summarize these observations as follows:

Theorem. An $n \times n$ matrix A is diagonalizable if and only if $\mathbb{R}^n$ has a basis $v_1, v_2, \ldots, v_n$ whose elements are all eigenvectors of A. In this case, if λ_i is the eigenvalue such that $Av_i = \lambda_i v_i$, then $\boxed{A = PDP^{-1}}$ for

$$P = \left[\begin{array}{cccc} v_1 & v_2 & \cdots & v_n \end{array}\right] \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

Prev slide said: if you can write $A = PDP^{-1}$ (D diagonal) then diag entries of D give eigvals of A and columns of P give eigvectors for A

This slide says: if you can find n linearly independent eigenvectors for A, then can find $A = PDP^{-1}$

this is only diagonal matrix w/ same size as B and same eigvals

$\boxed{\text{Not every matrix is diagonalizable.}}$

Example. The 2×2 matrix $B = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ has only one eigenvalue 2.

We saw above that B is not similar to $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, so B **is not diagonalizable**.

Theorem. If A is $n \times n$ with n distinct eigenvalues then A is diagonalizable.

Proof. Suppose A has n distinct eigenvalues.

Any choice of eigenvectors for A for these eigenvalues will be linearly independent.

Therefore A has n linearly independent eigenvectors.

These n linearly independent eigenvectors are a basis for $\mathbb{R}^n$ since $\dim(\mathbb{R}^n) = n$. $\square$

Example. The matrix $A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$ is triangular so has eigenvalues $5, 0, -2$.

These are distinct numbers, so A is diagonalizable.

Example. Not all diagonalizable $n \times n$ matrices have n distinct eigenvalues.

The identity matrix I is diagonalizable, but only has one distinct eigenvalue.

$$I = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

both diagonal and diagonalizable

but only 1 eigenvalue

($\lambda = 1$)

$f_2 = 0+1 = 1$, $f_3 = 1+1 = 2$, $f_4 = 1+2 = 3$, $f_5 = 2+3 = 5$

2 Diagonalization and Fibonacci numbers

Diagonalization leads to an exact formula for the *Fibonacci numbers*.

The sequence f_n of Fibonacci numbers starts as

$$f_0 = 0, \quad f_1 = 1, \quad f_2 = 1, \quad f_3 = 2, \quad f_4 = 3, \quad f_5 = 5, \quad f_6 = 8, \quad f_7 = 13 \quad \ldots$$

For $n \geq 2$, the sequence is defined by $\boxed{f_n = f_{n-2} + f_{n-1}.}$ "linear recurrence"

We have $f_{10} = 55$ and $f_{100} = 354224848179261915075$.

$$\begin{aligned} f_{n+1} &= f_{n-1} + f_n \\ f_{n+2} &= f_n + f_{n+1} \\ &= f_{n-1} + 2f_n \end{aligned}$$

Define $\boxed{a_n = f_{2n}}$ and $\boxed{b_n = f_{2n+1}}$ for $n \geq 0$.

If $n > 0$ then $a_n = f_{2n} = f_{2n-2} + f_{2n-1} = a_{n-1} + b_{n-1}$.

Similarly, if $n > 0$ then $b_n = f_{2n+1} = f_{2n-1} + f_{2n} = b_{n-1} + a_n = a_{n-1} + 2b_{n-1}$.

the definition of f_n means that we can compute

$$\begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix}$$

$a_n =$ (even indexed Fibonacci numbers) $= f_{2n}$

$b_n =$ (odd indexed Fibonacci numbers) $= f_{2n+1}$

$$f_n = f_{n-1} + f_{n-2} \implies a_n = f_{2n} = f_{2n-2} + f_{2n-1}$$
$$= a_{n-1} + b_{n-1}$$

$$\begin{cases} a_n = a_{n-1} + b_{n-1} \\ b_n = a_{n-1} + 2b_{n-1} \end{cases}$$

$$b_n = f_{2n+1} = f_{2n-1} + f_{2n}$$
$$= a_n + b_{n-1}$$
$$= a_{n-1} + 2b_{n-1}$$

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix}$$

$(a_n = f_{2n},\ b_n = f_{2n+1})$ $\begin{bmatrix} a_0 \\ b_0 \end{bmatrix} = \begin{bmatrix} f_0 \\ f_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

We can put these two equations together into one matrix equation:

$$\boxed{\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix}.}$$

Since this holds for all $n > 0$, we have

$$\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ b_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^2 \begin{bmatrix} a_{n-2} \\ b_{n-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^3 \begin{bmatrix} a_{n-3} \\ b_{n-3} \end{bmatrix} = \cdots = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^n \begin{bmatrix} a_0 \\ b_0 \end{bmatrix}.$$

In other words $\boxed{\begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}.}$

Thus if we could get an exact formula for the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^n$ then we could derive a formula for $a_n = f_{2n}$ and $b_n = f_{2n+1}$, which would determine f_n for all n.

One way to compute A^n for large values of n is to *diagonalize* A, as then can write

$$A = PDP^{-1} \quad \text{and} \quad \boxed{A^n = PD^nP^{-1}.}$$

$\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}^k = \begin{bmatrix} d_1^k & & \\ & \ddots & \\ & & d_n^k \end{bmatrix}$

D^n has simple formula

New goal: diagonalize

Steps: ① factor $\det(A - xI)$ to 2 distinct eigvals α and β

② because A is 2x2, it will be diagonalizable

just need to find nonzero vectors

$v \in \mathrm{Nul}(A - \alpha I)$

$w \in \mathrm{Nul}(A - \beta I)$

③ then $A = PDP^{-1}$

for $P = [v \ w]$

$D = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$

From this point on we let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$

①

To determine if A is diagonalizable, our first step is to compute its eigenvalues:

$$0 = \det(A - xI) = \det \begin{bmatrix} 1 - x & 1 \\ 1 & 2 - x \end{bmatrix} = (1 - x)(2 - x) - 1 = \boxed{x^2 - 3x + 1}$$

By the quadratic formula, the eigenvalues of A are

$$\boxed{\alpha = \frac{3 + \sqrt{5}}{2}} \text{ and } \boxed{\beta = \frac{3 - \sqrt{5}}{2}}.$$

Since $\alpha - \beta = \sqrt{5} \neq 0$, these eigenvalues are distinct so A is diagonalizable.

Our next step is to find bases for the α- and β-eigenspaces of A.

Continue to let $\boxed{A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}}$ along with $\boxed{\alpha = \frac{3+\sqrt{5}}{2}}$ and $\boxed{\beta = \frac{3-\sqrt{5}}{2}}$.

* If M is any square matrix and λ is any number, then we have a standard algorithm to compute a basis for $\mathrm{Nul}(M - \lambda I)$.

2

Using this algorithm, one can show that the α- and β-eigenspaces of A (that is, the null spaces of $A - \alpha I$ and $A - \beta I$) are both 1-dimensional with bases

$$\boxed{v = \begin{bmatrix} \alpha - 2 \\ 1 \end{bmatrix}} \quad \text{and} \quad \boxed{w = \begin{bmatrix} \beta - 2 \\ 1 \end{bmatrix}}$$

These vectors satisfy $\underline{Av = \alpha v}$ And $\underline{Bw = \beta w}$.

The details of how you row reduce $A - \alpha I$ and $A - \beta I$ to find these vectors are shown in the lecture notes.

* The computations are just a little more complicated than usual since α and β are numbers involving square roots rather than being integers or rational numbers.

Summary: we know that

$$\begin{bmatrix} f_{2n} \\ f_{2n+1} \end{bmatrix} = \begin{bmatrix} a_n \\ b_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \overbrace{P \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} P^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}^{\text{just need to simplify this}}$$

since $\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = PDP^{-1}$ for $D = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$

$$P = \begin{bmatrix} v & w \end{bmatrix} = \begin{bmatrix} \alpha-2 & \beta-2 \\ 1 & 1 \end{bmatrix}$$

where $\alpha = \frac{3+\sqrt{5}}{2}$ and $\beta = \frac{3-\sqrt{5}}{2}$

$$\left(\alpha = \frac{3+\sqrt{5}}{2}\right)\left(\beta = \frac{3-\sqrt{5}}{2}\right)$$

Since α and β are distinct eigenvalues of the 2×2 matrix A, with eigenvectors v and w, we know that A is diagonalizable and more specifically that $A = PDP^{-1}$ for

$$\boxed{P \stackrel{\text{def}}{=} \begin{bmatrix} v & w \end{bmatrix} = \begin{bmatrix} \alpha - 2 & \beta - 2 \\ 1 & 1 \end{bmatrix}} \quad \text{and} \quad \boxed{D \stackrel{\text{def}}{=} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}}.$$

Since P is 2×2 with $\det P = (\alpha - 2) - (\beta - 2) = \alpha - \beta = \sqrt{5}$, we have

$$\boxed{D^n = \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix}} \quad \text{and} \quad \boxed{P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 - \beta \\ -1 & \alpha - 2 \end{bmatrix}}. \quad = \frac{1}{\det P}\begin{bmatrix} 1 & 2-\beta \\ -1 & \alpha-2 \end{bmatrix}$$

We therefore have

$$\begin{aligned}
\begin{bmatrix} f_{2n} \\ f_{2n+1} \end{bmatrix} = \begin{bmatrix} a_n \\ b_n \end{bmatrix} &= A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= PD^nP^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha - 2 & \beta - 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} \begin{bmatrix} 1 & 2 - \beta \\ -1 & \alpha - 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

surprising amount of simplification ...

After multiplying out this product and making various simplifications, one gets

$$\begin{bmatrix} f_{2n} \\ f_{2n+1} \end{bmatrix} = PD^nP^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} \alpha^n - \beta^n \\ (\alpha-1)\alpha^n - (\beta-1)\beta^n \end{bmatrix}.$$

We now make one more unexpected simplification: because

$$(\alpha-1)^2 = \left(\tfrac{1+\sqrt{5}}{2}\right)^2 = \tfrac{1+2\sqrt{5}+5}{4} = \tfrac{3+\sqrt{5}}{2} = \alpha$$

$(\alpha-1)^2 = \alpha$

and

$$(\beta-1)^2 = \left(\tfrac{1-\sqrt{5}}{2}\right)^2 = \tfrac{1-2\sqrt{5}+5}{4} = \tfrac{3-\sqrt{5}}{2} = \beta$$

$(\beta-1)^2 = \beta$

we can rewrite the vector equation above as

$$\begin{bmatrix} f_{2n} \\ f_{2n+1} \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} (\alpha-1)^{2n} - (\beta-1)^{2n} \\ (\alpha-1)^{2n+1} - (\beta-1)^{2n+1} \end{bmatrix}.$$

This identity actually gives a single formula for f_n for all $n \geq 0$:

$$\boxed{f_n = \tfrac{1}{\sqrt{5}}\left((\alpha-1)^n - (\beta-1)^n\right)}.$$

rewrite

makes sense as α and β are solutions of $x^2-3x+1=0$ $\Leftrightarrow (x-1)^2 = x$

$\alpha = \frac{3+\sqrt{5}}{2}$ so $\alpha - 1 = \frac{1+\sqrt{5}}{2}$

likewise

$\beta - 1 = \frac{1-\sqrt{5}}{2}$

Summarizing everything, we conclude that:

Theorem. For all integers $n \geq 0$ it holds that

$$\boxed{f_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) \approx 0.447\,(1.618^n - (-0.618)^n)}$$

In fact, when n is large (like when $n > 10$), the term $(-0.618)^n \approx 0$.

Thus, a good approximation for f_n is the simple exponentiation function

$$\boxed{f_n \approx 0.447 \cdot 1.618^n}.$$

General principle:

f_n satisfies a 2-term recurrence

if you have a sequence s_n defined by a k-term linear recurrence

$$s_{n+k} = a_0 s_n + a_1 s_{n+1} + a_2 s_{n+2} + \cdots + a_{k-1} s_{n+k-1}$$

where $a_0, a_1, \ldots, a_{k-1} \in \mathbb{R}$ are constants

then you can write

$$\begin{bmatrix} a_{n+1} \\ a_{n+2} \\ a_{n+3} \\ \vdots \\ a_{n+k} \end{bmatrix} = \begin{bmatrix} \text{Some } k \times k \\ \text{matrix } A \end{bmatrix} \begin{bmatrix} a_{n-k+1} \\ \vdots \\ a_{n-2} \\ a_{n-1} \\ a_n \end{bmatrix}$$

if you can diagonalize A, then you can get a formula for s_n

3 Matrices with repeated eigenvalues

Suppose A is $n \times n$ and diagonalizable.

Then there exists an invertible $n \times n$ matrix P and a diagonal $n \times n$ matrix D with

$$A = PDP^{-1}.$$

If A has n distinct eigenvalues with corresponding eigenvectors $v_1, v_2, \dots, v_n$, then an easy way to construct such a matrix P is to just form $P = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$.

How do we find P if A does not have n distinct eigenvalues?

Theorem to remember:

$(p \leq n)$

The *multiplicity* of λ is the largest $m \geq 0$ such that $(\lambda - x)^m$ divides $\det(A - xI)$.

Theorem. Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$. Then:

(a) The dimension of $\mathrm{Nul}(A - \lambda_i I)$ is at most the multiplicity of λ_i.

(b) A is diagonalizable if and only if sum of dim of eig spaces = # columns of A

$$\dim \mathrm{Nul}(A - \lambda_1 I) + \dim \mathrm{Nul}(A - \lambda_2 I) + \cdots + \dim \mathrm{Nul}(A - \lambda_p I) = n. \quad (*)$$

(c) Suppose (*) holds and $\mathcal{B}_i$ is a basis for the λ_i-eigenspace.

The union $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_p$ is a basis for $\mathbb{R}^n$ consisting of eigenvectors of A.

If the elements of this union are the vectors $v_1, v_2, \ldots, v_n$ then the matrix

$$\boxed{P = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix}}$$

is invertible and the matrix $D = P^{-1}AP$ is diagonal, and $\boxed{A = PDP^{-1}}$.

$[a_1, a_2, \ldots, a_n] = [\lambda_1, \lambda_2, \ldots, \lambda_p]$

$D = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix}$ where $Av_i = a_i v_i$

has 2 eigvals :

5 and -3

Example. Consider the lower-triangular matrix

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}.$$

Its characteristic polynomial is $\det(A - xI) = (5-x)^2(-3-x)^2$.

The eigenvalues of A are therefore 5 and -3, each with *multiplicity* 2. Since

$$A - 5I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 4 & -8 & 0 \\ -1 & -2 & 0 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 8 & 16 \\ 0 & 1 & -4 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(A - 5I)$$

2 non-pivot cols

$\Rightarrow$ dim Nul$(A-5I) = 2$

it follows that $x \in \mathrm{Nul}(A - 5I)$ if and only if

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -8x_3 - 16x_4 \\ 4x_3 + 4x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

so

$$\boxed{\begin{bmatrix} -8 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -16 \\ 4 \\ 0 \\ 1 \end{bmatrix} \text{ is a basis for } \mathrm{Nul}(A - 5I).}$$

here is a basis

2 non-pivot columns

$\Rightarrow$ dim Nul$(A+3I)=2$

Since

$$A-(-3)I = A+3I = \begin{bmatrix} 8 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ -1 & -2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathsf{RREF}(A+3I)$$

it follows that $x \in \mathrm{Nul}(A+3I)$ if and only if

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so

$$\boxed{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{is a basis for } \mathrm{Nul}(A+3I).}$$

here is a basis

Each eigenspace has dimension 2, and $\boxed{2+2=4=n.}$

Thus A is diagonalizable. In particular, if

$$P = \begin{bmatrix} -8 & -16 & 0 & 0 \\ 4 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

then $\boxed{A = PDP^{-1}}$

basis for (-3)-eigspace

basis for
5-eigspace

4 A property of the Fibonacci sequence

The first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$ If we add up all the decimal numbers

$$\begin{array}{l} 0.0 \\ 0.01 \\ 0.001 \\ 0.0002 \\ 0.00003 \\ 0.000005 \\ 0.0000008 \\ 0.00000013 \\ 0.000000021 \\ 0.0000000034 \\ 0.00000000055 \\ 0.000000000089 \\ \vdots \end{array}$$

then we get exactly $1/89 = 0.011235955056179\cdots$.

More precisely:

$$\boxed{\frac{1}{89} = \sum_{n=0}^{\infty} \frac{f_n}{10^{n+1}}.}$$

Proof. If $x \neq 1$ then $\sum_{n=0}^{N-1} x^n = \frac{1-x^N}{1-x}$ since

$$(1-x)\sum_{n=0}^{N-1} x^n = (1 + x + x^2 + \cdots + x^{N-1}) - (x + x^2 + x^3 + \cdots + x^N) = 1 - x^N.$$

If $|x| < 1$ so that $x^N \to 0$ as $N \to \infty$ then $\sum_{n=0}^{\infty} x^n = \lim_{N\to\infty} \sum_{n=0}^{N} x^n = \frac{1}{1-x}$.

Now remember that $\sum_{n=0}^{\infty} \frac{f_n}{10^{n+1}} = \frac{1}{10\sqrt{5}} \sum_{n=0}^{\infty} \left(\left(\frac{1+\sqrt{5}}{20} \right)^n - \left(\frac{1-\sqrt{5}}{20} \right)^n \right)$.

We have both $|\frac{1+\sqrt{5}}{20}| < 1$ and $|\frac{1-\sqrt{5}}{20}| < 1$ so

$$\sum_{n=0}^{\infty} \left(\left(\frac{1+\sqrt{5}}{20} \right)^n - \left(\frac{1-\sqrt{5}}{20} \right)^n \right) = \sum_{n=0}^{\infty} \left(\frac{1+\sqrt{5}}{20} \right)^n - \sum_{n=0}^{\infty} \left(\frac{1-\sqrt{5}}{20} \right)^n = \frac{1}{1 - \frac{1+\sqrt{5}}{20}} - \frac{1}{1 - \frac{1-\sqrt{5}}{20}}.$$

The last expression can be simplified a lot:

$$\frac{1}{1 - \frac{1+\sqrt{5}}{20}} - \frac{1}{1 - \frac{1-\sqrt{5}}{20}} = \frac{20}{19 - \sqrt{5}} - \frac{20}{19 + \sqrt{5}} = \frac{20(19 + \sqrt{5}) - 20(19 - \sqrt{5})}{(19 - \sqrt{5})(19 + \sqrt{5})} = \frac{10\sqrt{5}}{89}.$$

Substituting this above gives

$$\sum_{n=0}^{\infty} \frac{f_n}{10^{n+1}} = \frac{1}{10\sqrt{5}} \sum_{n=0}^{\infty} \left(\left(\frac{1+\sqrt{5}}{20} \right)^n - \left(\frac{1-\sqrt{5}}{20} \right)^n \right) = \frac{1}{10\sqrt{5}} \frac{10\sqrt{5}}{89} = \frac{1}{89}.$$

$\square$