MATH 2121 — Lecture #17

Today:

- Review: diagonalizable matrices
- New: self-contained introduction to complex numbers

diagonalization of A
(or the eigenval decomp)

1 Last time: methods to check diagonalizability

Let n be a positive integer and let A be an $n \times n$ matrix.

similar to diag matrix

Remember that A is *diagonalizable* if $\boxed{A = PDP^{-1}}$ where P is an invertible $n \times n$ matrix and D is an $n \times n$ diagonal matrix.

Suppose $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ is a basis and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are numbers. Define

$$P = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}.$$

If $A = PDP^{-1}$ then $Av_i = PDP^{-1}v_i = PDe_i = \lambda_i Pe_i = \lambda_i v_i$ for each $i = 1, 2, \ldots, n$.

When $A = PDP^{-1}$, the columns of P are a basis for $\mathbb{R}^n$ of eigenvectors of A.

If $A = PDP^{-1}$ then $\begin{cases} \text{diag entries of } D = \text{eigvals of } A \\ \text{columns of } P = \text{bases for each eigenspace of } A \end{cases}$

• "most" matrices are diagonalizable, but not all

Matrices that are not diagonalizable.

Proposition. Let A be an $n \times n$ upper-triangular matrix with all entries on the diagonal equal to λ:

$$A = \begin{bmatrix} \lambda & * & \cdots & * \\ & \lambda & \ddots & \vdots \\ & & \ddots & * \\ & & & \lambda \end{bmatrix}.$$

← upper △ matrix with same entry in all diagonal entries

(The entries above the diagonal can be any numbers.)

$\boxed{\text{If } A \text{ is not the diagonal matrix } \lambda I, \text{ then } A \text{ is not diagonalizable.}}$

Proof. Suppose $A = PDP^{-1}$ where D is diagonal.

Every diagonal entry of D is an eigenvalue for A.

Only eigenvalue of A is λ so $\boxed{D = \lambda I}$ and $A = \underline{P(\lambda I)P^{-1} = \lambda PIP^{-1}} = \lambda I$. ☐

is of our special form

so if A and diagonalizable then A is diagonal

$PIP^{-1} = PP^{-1} = I$

Diagonalizable: $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ (why? it's diagonal)

Not diagonalizable: $\begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

this can't be diagonalizable as its only eigenvalue is $\lambda = 3$ but it's not similar to $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

observe: $\begin{bmatrix} 3 & & \\ & 3 & \\ & & 3 \end{bmatrix} = 3I$

is only similar to itself:

$$P\,3I\,P^{-1} = 3\cancel{P}\cancel{I}\cancel{P^{-1}} = 3I$$

The following result summarizes everything we need to know about diagonalizability: how to determine if a matrix A is diagonalizable, and then how to compute the decomposition $\boxed{A = PDP^{-1}}$ if it exists.

Theorem. Let A be an $n \times n$ matrix.

Suppose $\lambda_1, \lambda_2, \ldots, \lambda_p$ are the distinct eigenvalues of A.

find by solving $\det(A - xI) = 0$

Let $\boxed{d_i = \dim \mathrm{Nul}(A - \lambda_i I)}$ for $i = 1, 2, \ldots, p$.

By the definition of an eigenvalue, we have $\boxed{1 \le d_i \le n}$ for each i. Moreover:

1. We always have $d_1 + d_2 + \cdots + d_p \le n$.
2. The matrix $\boxed{A \text{ is diagonalizable if and only if } d_1 + d_2 + \cdots + d_p = n.}$

not $\begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}$ so $\mathrm{Nul} \neq \mathbb{R}^3$

d_i = # non-pivot columns in $\mathrm{RREF}(A - \lambda_i I)$

Note: this means if $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ then $\dim \mathrm{Nul}(A - 3I) < 3$

NOT diagonalizable, has only one eigval $\lambda = 3$

so $d_1 + d_2 + \dots + d_p = n$

3. Suppose A is diagonalizable. Let $D_i = \lambda_i I_{d_i}$. Let D be $n \times n$ diagonal matrix

$d_2 \times d_2$ block

$$D = \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_p \end{bmatrix}$$

$d_p \times d_p$ block

Choose n vectors $v_1, v_2, \dots, v_n \in \mathbb{R}^n$ such that

- the first d_1 vectors are a basis for $\mathrm{Nul}(A - \lambda_1 I)$,
- the next d_2 vectors are a basis for $\mathrm{Nul}(A - \lambda_2 I)$,
- the next d_3 vectors are a basis for $\mathrm{Nul}(A - \lambda_3 I)$,

  $\vdots$

- the last d_p vectors are basis for $\mathrm{Nul}(A - \lambda_p I)$.

Then $\underbrace{A = PDP^{-1}}$ for $\boxed{P = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}.}$

this will automatically hold

if $d_1 = 1$, $d_2 = 3$, $d_3 = 2$

then

$$D = \begin{bmatrix} \lambda_1 & & & & & \\ & \lambda_2 & & & & \\ & & \lambda_2 & & & \\ & & & \lambda_2 & & \\ & & & & \lambda_3 & \\ & & & & & \lambda_3 \end{bmatrix}$$

Ex What's a matrix that is not invertible or diagonalizable?

$$\begin{bmatrix} 0 & -6 \\ 0 & 0 \end{bmatrix}$$ (det = 0 so not invertible)
(not diagonalizable)

Ex What's a matrix that is not invertible but is diagonalizable?

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ (not invertible)
(but it's diagonal and so diagonalizable)

Ex What's a matrix that is not invertible, not diagonalizable, but has more than one eigenvalue?

$$\left[\begin{array}{cc|c} 0 & -6 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$$

has eigvals 0, 1, not invertible, also not diagonalizable

Next: self-contained introduction to $\boxed{\text{complex numbers}}$

idea: the complex numbers are the smallest way of enlarging the set of real numbers so that polynomial equations (like $\underbrace{x^2+1=0}_{\text{has no solutions with } x\in\mathbb{R}}$) always have solutions

$\boxed{\text{Concretely: a complex number is just a kind of } 2\times2 \text{ matrix}}$

Notation: define $i \overset{\text{def}}{=} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (recall: this is a 90° rotation matrix)

2 Complex numbers

Also: $a+bi \overset{\text{def}}{=} a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

For the rest of this lecture, let $i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Recall that $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Suppose $a, b \in \mathbb{R}$. Both i and I_2 are 2×2 matrices, so we can form the sum $\boxed{aI_2 + bi.}$ $(a, b \in \mathbb{R})$

To simplify our notation, we will write 1 instead of I_2 and $\boxed{a + bi}$ instead of $aI_2 + bi$.

We consider $a = a + 0i$ and $bi = 0 + bi$ and $0 = 0 + 0i$. With this convention

$$a + bi = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Define $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\} = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$.

This is called the set of *complex numbers*.

Each element of $\mathbb{C}$ is a 2×2 matrix, to be called a *complex number*.

arithmetic in $\mathbb{C}$ is just a special case of matrix operations

Notational conventions $\quad a, b, c, d \in \mathbb{R}$

$$a+bi = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$0+bi = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix} = bi$$

2x2 identity matrix

$$a+0i = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = aI = a$$

skip I to save time

$$0+0i = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

arithmetic in $\mathbb{C}$: $+$ $-$ $\times$ $\div$ $(\dots)^{-1}$

commutative

Fact. We can add complex numbers together. If $a, b, c, d \in \mathbb{R}$ then

$$(a+bi)+(c+di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a+c & -b-d \\ b+d & a+c \end{bmatrix} = (a+c)+(b+d)i \in \mathbb{C}.$$

Clearly $\boxed{(a+bi)+(c+di) = (c+di)+(a+bi) = (a+c)+(b+d)i}$.

Fact. We can subtract complex numbers. If $a, b, c, d \in \mathbb{R}$ then

$$(a+bi)-(c+di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} - \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a-c & -b+d \\ b-d & a-c \end{bmatrix} = \underline{(a-c)+(b-d)i \in \mathbb{C}}.$$

$$(a+bi)+(c+di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a+c & -b-d \\ b+d & a+c \end{bmatrix}$$

$$= (a+c)+(b+d)i \in \mathbb{C}$$

$$\underbrace{(5+7i) + (-1+2i)}_{\text{also written as}} = \underbrace{4+9i}_{\text{also written as}}$$

$$\begin{bmatrix} 5 & -7 \\ 7 & 5 \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -9 \\ 9 & 4 \end{bmatrix}$$

consider as same

$$(5+7i) + (5-7i) = 10 + 0i = 10 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

Nontrivial fact

$(a+bi)(c+di) = (c+di)(a+bi)$

mult. is $\underbrace{\text{commutative}}$ in $\mathbb{C}$, though not in $\mathbb{R}^{2\times 2}$

Fact. We can multiply complex numbers. If $a, b, c, d \in \mathbb{R}$ then

$$(a+bi)(c+di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix} = (ac-bd)+(ad+bc)i \in \mathbb{C}.$$

Note that $\boxed{(a+bi)(c+di) = (c+di)(a+bi) = (ac-bd)+(ad+bc)i}$.

Fact. We can multiply complex numbers by real numbers. If $a, b, x \in \mathbb{R}$ then define

$$(a+bi)x = x(a+bi) = x\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} ax & -bx \\ bx & ax \end{bmatrix} = (ax)+(bx)i \in \mathbb{C}.$$

$(a+bi)(x+0i)$

Note that this is the same as the product $(a+bi)(x+0i)$.

equal diag entries

$$\underbrace{(a+bi)(c+di)}_{\in\,\mathbb{C}} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -ad-bc \\ ad+bc & ac-bd \end{bmatrix}$$

opposite off-diag entries

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

different

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

not complex numbers

$$\overset{-i}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}\overset{i}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 = (-i)i$$

are complex numbers

$$\underset{i}{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}\underset{-i}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 = i(-i)$$

equal

define division in $\mathbb{C}$: $\dfrac{a+bi}{x}$ $= x^{-1} \cdot (a+bi) = (a+bi) \cdot x^{-1}$

(by numbers in $\mathbb{R}$) $(x \in \mathbb{R})$ $(x \neq 0)$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x^{-1} & 0 \\ 0 & x^{-1} \end{bmatrix} = \begin{bmatrix} ax^{-1} & -bx^{-1} \\ bx^{-1} & ax^{-1} \end{bmatrix} = \frac{a}{x} + \frac{b}{x} i$$

Fact. We can divide complex numbers by nonzero real numbers.

If $a, b, x \in \mathbb{R}$ and $x \neq 0$ then define

$$(a+bi)/x = (a+bi)(1/x) = (a/x) + (b/x)i.$$

We sometimes write $\boxed{\frac{p}{q} \text{ instead of } p/q.}$ Both expressions means the same thing.

A complex number $a + bi$ is *nonzero* if $a \neq 0$ or $b \neq 0$. Since

$$\boxed{\det(a+bi) = \det \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a^2 + b^2,}$$

which is only zero if $a = b = 0$, every nonzero complex number is invertible. (has det $\neq 0$)

$a+bi \neq 0$ means $\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\iff (a \neq 0$ or $b \neq 0)$

Rmk We can add elems of $\mathbb{C}$ and multiply them by elems of $\mathbb{R}$ (and stay inside the set $\mathbb{C}$)

$\Rightarrow$ $\mathbb{C}$ is a (real) vector space (a subspace of $\mathbb{R}^{2\times 2}$ = 2x2 matrices)

$\dim \mathbb{C} = 2$, one basis is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\dim \mathbb{R}^{2\times 2} = 4$, one basis is $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

idea for division: define $\frac{a+bi}{c+di} = (a+bi)(c+di)^{-1} = (c+di)^{-1}(a+bi)$

equal

(when $c+di \neq 0$)

Fact. We can divide complex numbers. If $a, b, c, d \in \mathbb{R}$ and $c + di \neq 0$ then define

$$(a+bi)/(c+di) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1}.$$

We can write this more explicitly as

$$\begin{aligned}
(a+bi)/(c+di) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} \\
&= \frac{1}{c^2+d^2} \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \\
&= \frac{1}{c^2+d^2} \begin{bmatrix} ac+bd & ad-bc \\ bc-ad & ac+bd \end{bmatrix} = \boxed{\frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2} i \in \mathbb{C}.}
\end{aligned}$$

The last formula is not so easy to remember.

but what is $(c+di)^{-1} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} = \frac{1}{c^2+d^2} \begin{bmatrix} c & d \\ -d & c \end{bmatrix}$

$= \frac{c}{c^2+d^2} - \frac{d}{c^2+d^2} i \in \mathbb{C}$

Shortcut: think of $a+bi$ as polynomial in a variable i that has $i^2=-1$ (and use polynomial ops)

It may be easier to divide complex numbers using the following method:

Example. We have

$$\frac{3-4i}{2+i} = \frac{(3-4i)(2-i)}{(2+i)(2-i)} = \frac{6-3i-8i+4i^2}{4-i^2} = \frac{6-11i-4}{5} = \frac{2-11i}{5} = \frac{2}{5} - \frac{11}{5}i.$$

$4-2i+2i-i^2$

More generally, if $c+di \neq 0$ then we always have $\boxed{\dfrac{a+bi}{c+di} = \dfrac{(a+bi)(c-di)}{c^2+d^2}}$ since

$$\frac{a+bi}{c+di} = (a+bi)(c+di)^{-1} = \frac{1}{c^2+d^2}(a+bi)(c-di) = \frac{(a+bi)(c-di)}{c^2+d^2}.$$

<u>Obs</u> $i^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1$

<u>Obs</u> $(c+di)(c-di) = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}\begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} c^2+d^2 & 0 \\ 0 & c^2+d^2 \end{bmatrix}$

$= (c^2+d^2)$ no i involved

Division in $\mathbb{C}$:

$$\boxed{\frac{a+bi}{c+di}} \overset{\text{def}}{=} (c+di)^{-1}(a+bi) = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

(must have $c \neq 0$ or $d \neq 0$)

$$\frac{1}{c^2+d^2}(a+bi)(c-di)$$

$$= \frac{1}{c^2+d^2} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

$$= \frac{1}{c^2+d^2} \begin{bmatrix} ac+bd & ad-bc \\ bc-ad & ac+bd \end{bmatrix}$$

$$= \boxed{\frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}\, i} \in \mathbb{C}$$

still in $\mathbb{C}$

(no need to memorize this formula)

The $\boxed{\textit{complex conjugate}}$ of $c+di$ is its matrix transpose, in other words:

$$\overline{c+di} = (c+di)^\top = c - di \in \mathbb{C}.$$

When $c+di$ is nonzero, the complex conjugate is related to the inverse by

$$(c+di)^{-1} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix}^{-1} = \frac{1}{c^2+d^2}\begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \frac{1}{c^2+d^2} \cdot \overline{c+di}.$$

Since $x, y \in \mathbb{C}$ satisfy $xy = yx$ and $(xy)^\top = y^\top x^\top$ (as matrices), it follows that

$$\boxed{\overline{xy} = \overline{y} \cdot \overline{x} = \overline{x} \cdot \overline{y}.}$$

also: $\overline{x+y} = \overline{x} + \overline{y}$

$$\overline{a+bi} \stackrel{\text{def}}{=} \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^\top = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = a - bi \in \mathbb{C}$$

$(a+bi)+c = (a+bi)+(c+0i) = (a+c)+di$

We can also add complex numbers $a+bi$ with real numbers c when $a, b, c \in \mathbb{R}$.

To do this, we set $c = c + 0i$ and define

$$(a+bi)+c = c+(a+bi) = (a+bi)+(c+0i) = (a+c)+bi.$$

Under this convention, we have

$$i^2+1 = (0+i)(0+i)+(1+0i) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0+0i = 0.$$

Thus it makes sense to write $\boxed{i^2 = -1}$.

means eq $x^2+1=0$ has solutions in $\mathbb{C}$

(namely $x = i$ or $-i$)

$$\boxed{e^{\pi i} + 1 = 0}$$

what does this mean?

① $e^{\pi i} = e^{\begin{bmatrix} 0 & -\pi \\ \pi & 0 \end{bmatrix}}$

② $e^x \overset{\text{def}}{=} 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^n}{24} + \dots$

Theorem. Define the exponential function $\mathbb{C} \to \mathbb{C}$ by the convergent power series

$$e^x = 1 + \frac{1}{1}x + \frac{1}{1 \cdot 2}x^2 + \frac{1}{1 \cdot 2 \cdot 3}x^3 + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \dots$$

③ $e \approx 2.7$

unique real number such that

$\frac{d}{dx} e^x = e^x$

Then $e^1 = e = 2.71828\dots$ and $\boxed{e^{i\pi} + 1 = 0}$.

Proof. We need two facts from calculus:

$$-1 = \cos \pi = 1 - \frac{1}{1 \cdot 2}\pi^2 + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4}\pi^4 - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}\pi^6 + \dots$$

$$0 = \sin \pi = \frac{1}{1}\pi - \frac{1}{1 \cdot 2 \cdot 3}\pi^3 + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}\pi^5 - \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}\pi^7 + \dots$$

We have

$$i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad i^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad i^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \text{and} \quad i^0 = i^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

i $\quad -1 \quad$ $-i$ $\quad 1$

Thus $i^{n+4} = i^n$ for all n.

Also, we have $(i\pi)^n = \pi^n i^n$. It follows that

$$e^{i\pi} = \begin{bmatrix} 1 - \frac{1}{1\cdot 2}\pi^2 + \frac{1}{1\cdot 2\cdot 3\cdot 4}\pi^4 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\pi^6 + \dots & \frac{1}{1}\pi - \frac{1}{1\cdot 2\cdot 3}\pi^3 + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\pi^5 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\pi^7 + \dots \\ \frac{1}{1}\pi - \frac{1}{1\cdot 2\cdot 3}\pi^3 + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5}\pi^5 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\pi^7 + \dots & 1 - \frac{1}{1\cdot 2}\pi^2 + \frac{1}{1\cdot 2\cdot 3\cdot 4}\pi^4 - \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\pi^6 + \dots \end{bmatrix}$$

By our two facts, this is just $e^{\overset{\pi i}{i\pi}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -1 + 0i$.

Thus $e^{\overset{\pi i}{i\pi}} + 1 = (-1 + 0i) + (1 + 0i) = 0$. ∎

After a while, we tend to view the elements of $\mathbb{C}$ as formal expression $\boxed{a + bi}$ where $a, b \in \mathbb{R}$ and i is a symbol that satisfies $\boxed{i^2 = -1.}$

We can add, subtract, and multiply such expressions just like polynomials, but substituting -1 for i^2. This convention gives the same operations as we saw above.

Moreover, this makes it clearer how to view $\mathbb{R}$ as a subset of $\mathbb{C}$, by setting $a = a + 0i$.

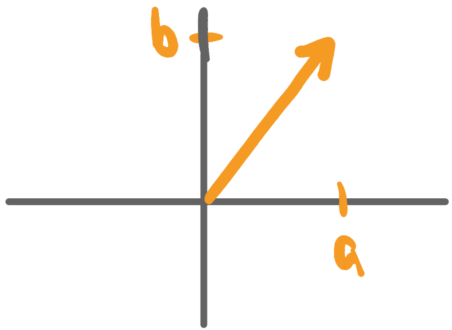
The *real part* of a complex number $a + bi \in \mathbb{C}$ is $\Re(a + bi) = \boxed{a \in \mathbb{R}.}$

The *imaginary part* of $a + bi \in \mathbb{C}$ is $\Im(a + bi) = \boxed{b \in \mathbb{R}.}$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Remark. It can be helpful to draw $a + bi \in \mathbb{C}$ as the vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$.

The number $i(a + bi) = -b + ai \in \mathbb{C}$ then corresponds to the vector $\begin{bmatrix} -b \\ a \end{bmatrix} \in \mathbb{R}^2$, which is given by *rotating* $\begin{bmatrix} a \\ b \end{bmatrix}$ *ninety degrees counterclockwise.*



We work with complex numbers because of the following theorem about *polynomials*.

Suppose

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is a polynomial with coefficients $a_0, a_1, \ldots, a_n \in \mathbb{C}$.

Assume $a_n \neq 0$ so that $p(x)$ has *degree* n.

This expression for $p(x)$ still makes sense for $x \in \mathbb{C}$.

Theorem (*Fundamental theorem of algebra*)**.** There are n (not necessarily distinct) complex numbers $r_1, r_2, \ldots, r_n \in \mathbb{C}$ such that

$$\boxed{p(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n).}$$

One calls the numbers $r_1, r_2, \ldots, r_n$ the *roots* of $p(x)$.

The roots of $p(x)$ give all solutions to the equation $\boxed{p(x) = 0.}$

A root r has *multiplicity* m if exactly m of the numbers $r_1, r_2, \ldots, r_n$ are equal to r.

The use of complex numbers in this theorem is essential.

The statement fails if we use $\mathbb{R}$ instead of $\mathbb{C}$.

Example: if $\boxed{p(x) = x^2 + 1}$ then there **do not exist** real numbers $r_1, r_2 \in \mathbb{R}$ with

$$p(x) = (x - r_1)(x - r_2).$$

However, we do have $\boxed{x^2 + 1 = (x - i)(x + i).}$