MATH 2121 – Lecture #20

Outline:

1. Review: inner products
   orthogonality

2. Orthogonal projections
   Orthonormal vectors
   Gram-Schmidt process

1 Last time: inner products and orthogonality

The $\boxed{\textit{inner product} \text{ or } \textit{dot product}}$ of vectors $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$ is

$$\boxed{u \bullet v = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n} = u^\top v = v^\top u = v \bullet u \in \mathbb{R}.$$

two vectors $\rightsquigarrow$ scalar

$u \bullet v = u^T v$

The *length* of a vector $v \in \mathbb{R}^n$ is $\|v\| = \sqrt{v \bullet v} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$.

A vector with length 1 is a *unit vector*. Note that $\|v\|^2 = v \bullet v$.

$(v \cdot v = 1)$

Two vectors $u, v \in \mathbb{R}^n$ are *orthogonal* if $u \bullet v = 0$.

* In $\mathbb{R}^2$, two vectors are orthogonal if and only if they belong to perpendicular lines.

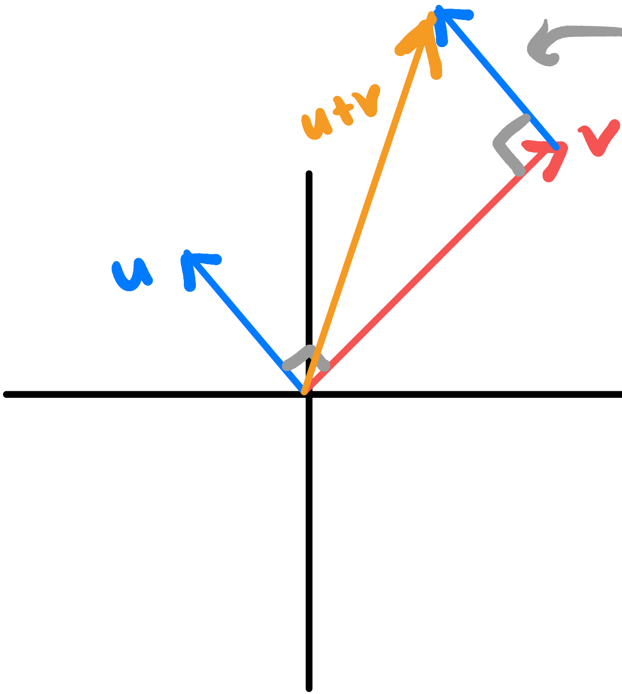
* **Pythagorean Theorem**: $u, v \in \mathbb{R}^n$ are orthogonal iff $\|u + v\|^2 = \|u\|^2 + \|v\|^2$.

Pythagorean thm

$a^2+b^2=c^2$ if

Inner product version: $\boxed{\|u+v\|^2 = \|u\|^2+\|v\|^2}$ iff $u\cdot v=0$

Picture:



right triangle with $\begin{cases} a = \|v\| \\ b = \|u\| \\ c = \|u+v\| \end{cases}$

$V^\perp = \{ w \in \mathbb{R}^n \mid v \bullet w = 0 \text{ for all } v \in V \}$

this is another subspace "V perp"

The $\boxed{\textit{orthogonal complement}}$ of a subspace $V \subseteq \mathbb{R}^n$ is the subspace $V^\perp$ whose elements are the vectors $w \in \mathbb{R}^n$ such that $w \bullet v = 0$ for all $v \in V$.

The only vector that is in both V and $V^\perp$ is the zero vector, so $\boxed{V \cap V^\perp = \{0\}}$.

(because $v \bullet v = 0 \Rightarrow v = 0$)

Ex We have $\{0\}^\perp = \mathbb{R}^n$ and $(\mathbb{R}^n)^\perp = \{0\}$.

* If A is an $m \times n$ matrix then $\boxed{(\mathrm{Col}\, A)^\perp = \mathrm{Nul}(A^\top)}$

because we can find bases for Null spaces, can also find a basis for $V^\perp$

* We also saw last time that $\boxed{\dim V + \dim V^\perp \leq n}$.

A list $u_1, u_2, \ldots, u_p \in \mathbb{R}^n$ is *orthogonal* if $u_i \bullet u_j = 0$ whenever $1 \leq i < j \leq p$.

Theorem. Any list of $\boxed{\text{orthogonal}}$ nonzero vectors is linearly independent and so is an orthogonal basis of the subspace it spans.

↳ if you have a lin. comb. $c_1 v_1 + \cdots + c_k v_k$ of orthogonal, nonzero vectors, can recover $c_1, c_2, \ldots, c_k$ by taking inner products

Another proof: suppose $u_1, u_2 \dots u_k$ are nonzero and orthogonal

Then $u_i \cdot u_i > 0$ (as $u_i \neq 0$)

$u_i \cdot u_j = 0$ if $i \neq j$

$\Rightarrow$ if $A = [u_1 \; u_2 \dots u_k]$ then

invertible diag.

$$A^T A = [u_i \cdot u_j]_{\substack{1 \le i \le k \\ 1 \le j \le k}} = \begin{bmatrix} u_1 \cdot u_1 & & & 0 \\ & u_2 \cdot u_2 & & \\ & & \ddots & \\ 0 & & & u_k \cdot u_k \end{bmatrix}$$

Thus if $Ax = 0$ then $x = (A^T A)^{-1} A^T A x = (A^T A)^{-1} A^T 0 = 0$ $\square$

$$\left[\begin{array}{cccc|c} u_1 & u_2 & \cdots & u_p & y \end{array}\right] \xrightarrow{\text{RREF}} \left[\begin{array}{cccc|c} 1 & & & & c_1 \\ & 1 & & & c_2 \\ & & \ddots & & \vdots \\ & & & 1 & c_p \end{array}\right]$$

If $u_1, u_2, \ldots, u_p$ is an orthogonal basis for a subspace $V \subseteq \mathbb{R}^n$ and $\boxed{y \in V,}$ then

$$y = c_1 u_2 + c_2 u_2 + \cdots + c_p u_p \qquad \boxed{\text{where } c_i = \frac{y \bullet u_i}{u_i \bullet u_i} \in \mathbb{R}.}$$

$(c_1\ c_2 \cdots c_p \in \mathbb{R})$

Usually, to determine the coefficients that express a vector in a given basis, we have to solve a $\boxed{\text{linear system}}$. For orthogonal bases, we can just compute inner products.

Example. The standard basis $e_1, e_2, \ldots, e_n$ for $\mathbb{R}^n$ is orthogonal.

Therefore if $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ then $y = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n$ where $c_i = \frac{y \bullet e_i}{e_i \bullet e_i}$.

But $e_i \bullet e_i = 1$ and $y \bullet e_i = y_i$, so we just have $c_i = y_i$, meaning that

$$y = y_1 e_1 + y_2 e_2 + \cdots + y_n e_n.$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = y_1 e_1 + y_2 e_2 + y_3 e_3$$

$\mathrm{proj}_L : \mathbb{R}^n \longrightarrow L$ (linear)

2 Orthogonal projection onto a line

Let $L \subseteq \mathbb{R}^n$ be a one-dimensional subspace. $\dim L = 1$

Then $\boxed{L = \mathbb{R}\text{-span}\{u\}}$ for any nonzero vector $u \in L$.

Let $y \in \mathbb{R}^n$. The *orthogonal projection* of y onto L is the vector

$$\boxed{\mathrm{proj}_L(y) = \frac{y \bullet u}{u \bullet u} u} \qquad \text{for any } 0 \neq u \in L.$$

if we replace u by cu then

$$\frac{y \cdot \cancel{c}u}{\cancel{c}u \cdot \cancel{c}u} \cancel{c}u = \frac{y \cdot u}{u \cdot u} u$$

The value of $\mathrm{proj}_L(y)$ does not depend on the choice of the nonzero vector u.

The *component of y orthogonal to L* is the vector $\boxed{z = y - \mathrm{proj}_L(y).}$

Proposition. The only vector $\cancel{x} \in L$ with $y - \cancel{x} \in L^\perp$ is $\cancel{x} = \mathrm{proj}_L(y)$.

The proof is some simple algebra: see the lecture notes.

alt. DEFINITION of $\mathrm{proj}_L(y)$

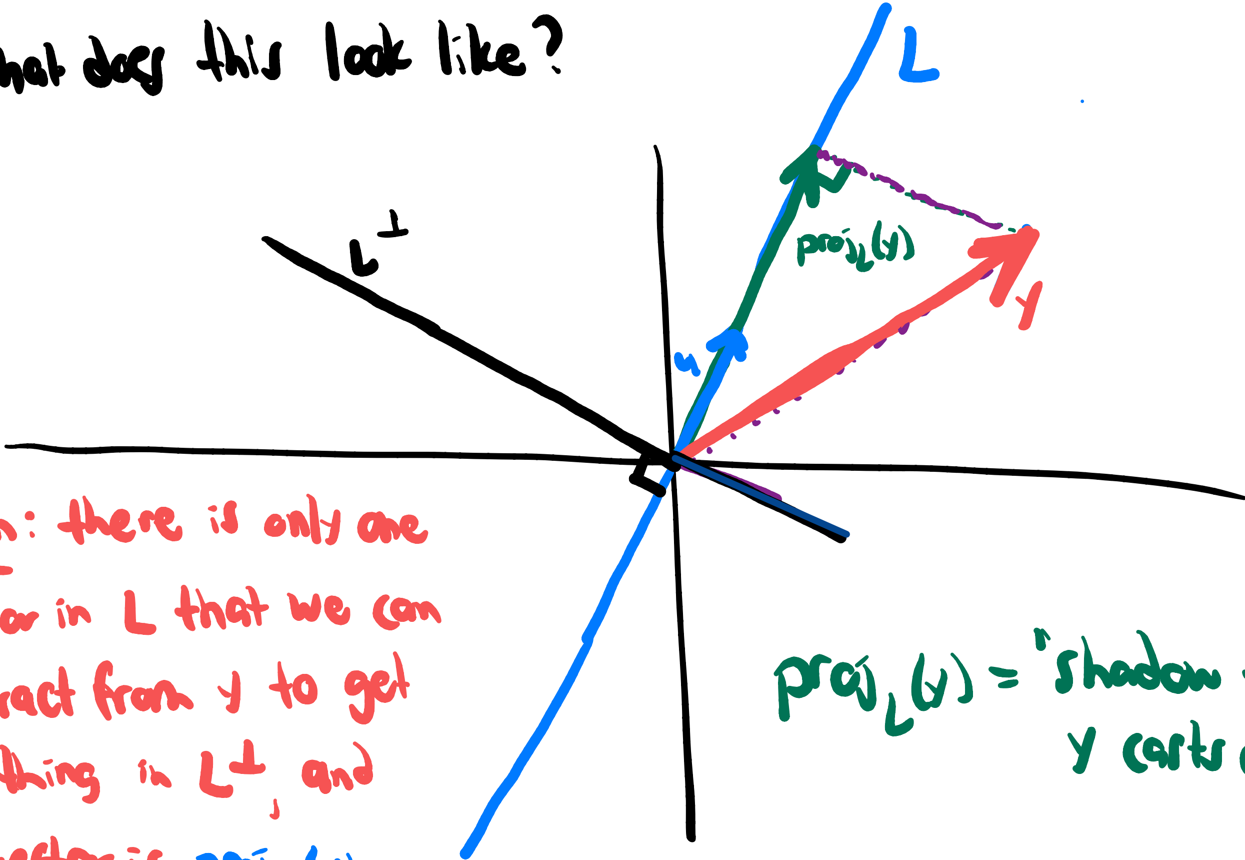
Check: $y - \mathrm{proj}_L(y) \in L^\perp$?

$$(y - \mathrm{proj}_L(y)) \cdot u$$

$$= \cancel{y \cdot u} - \frac{\cancel{y \cdot u}}{\cancel{u \cdot u}} \cancel{u \cdot u} = 0 \checkmark$$

$L = \mathbb{R}\text{-span}[u]$, $y \in \mathbb{R}^n$, $\text{proj}_L(y) = \frac{y \cdot u}{u \cdot u} u$, $z = y - \text{proj}_L(y)$

what does this look like?

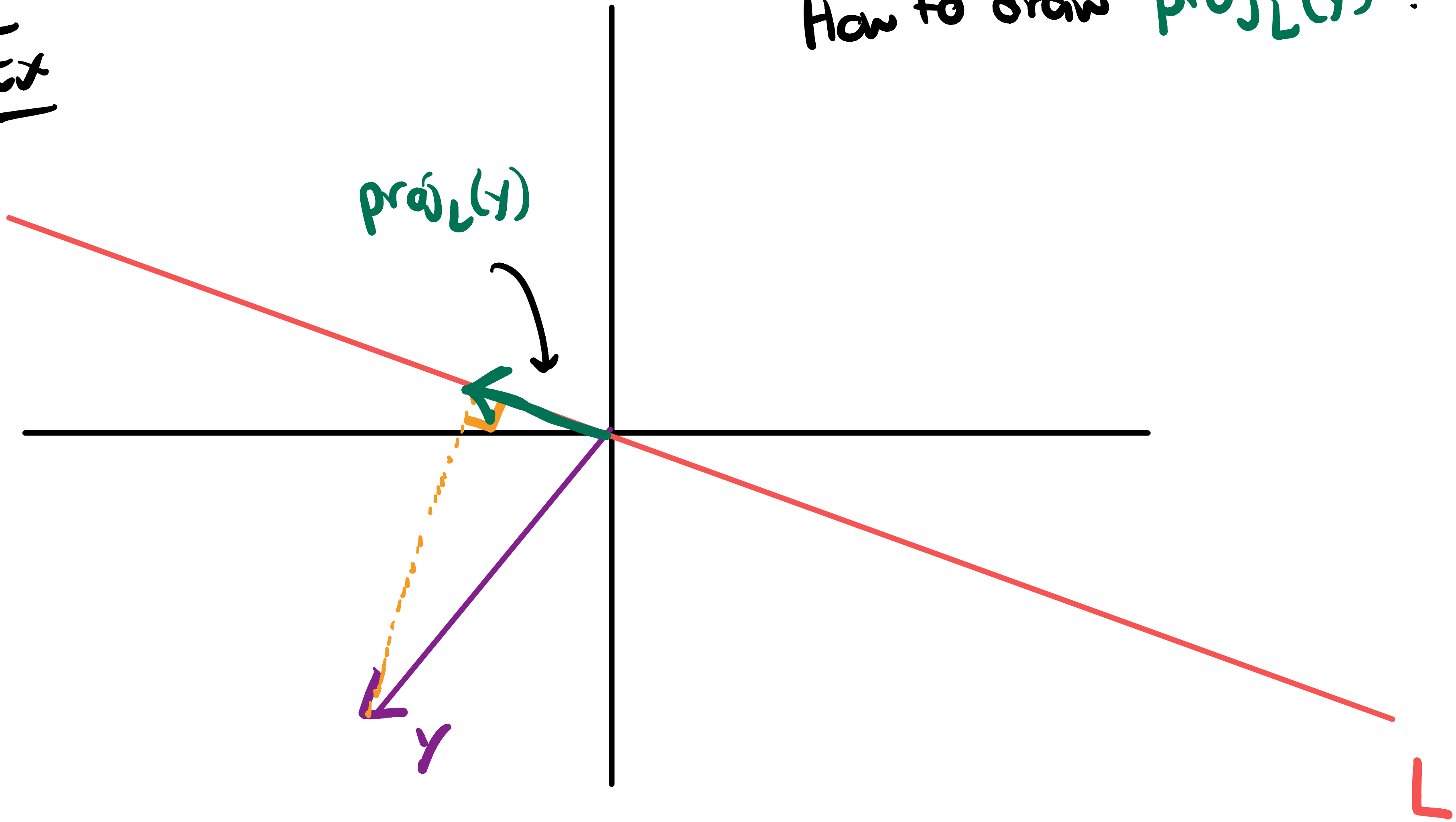


Claim: there is only one vector in L that we can subtract from y to get something in $L^\perp$, and this vector is $\text{proj}_L(y)$

$\text{proj}_L(y)$ = "shadow that y casts on L"

Ex

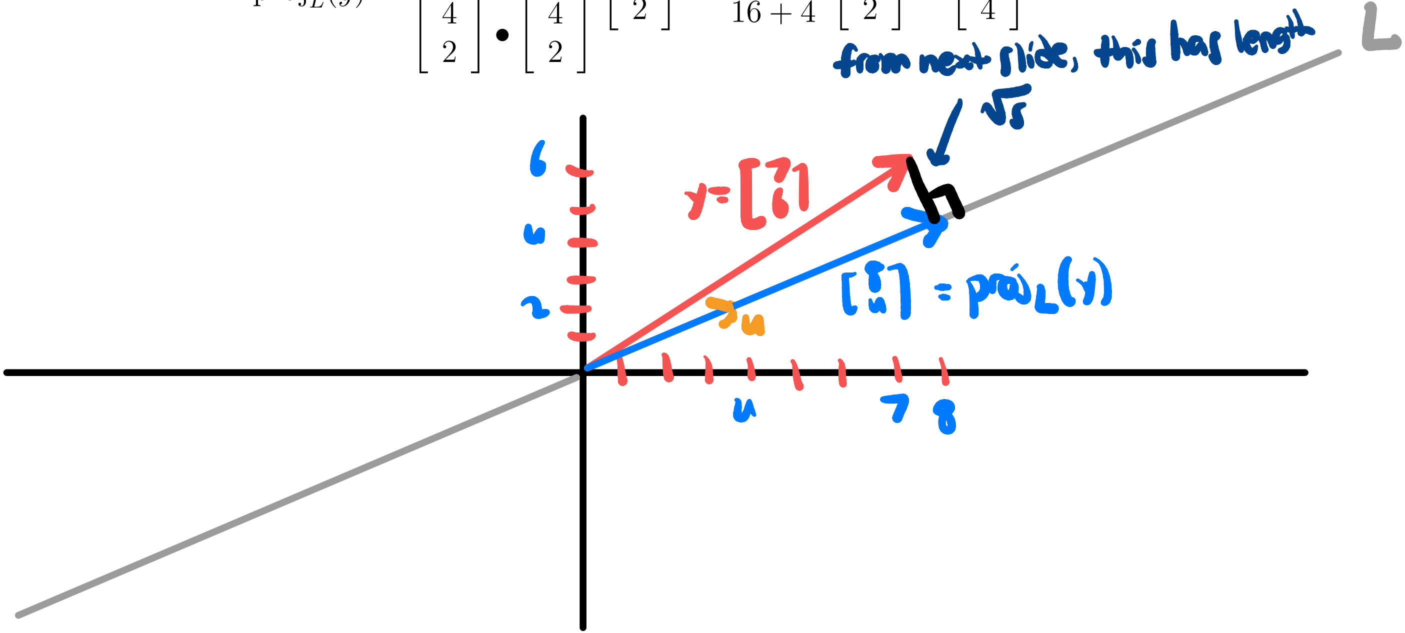
How to draw $proj_L(y)$?



Formula: $proj_L(y) = \frac{y \cdot u}{u \cdot u} u$ for any $0 \neq u \in L$

Example. If $y = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $L = \mathbb{R}\text{-span}\left\{ \overset{u}{\begin{bmatrix} 4 \\ 2 \end{bmatrix}} \right\}$ then

$$\operatorname{proj}_L(y) = \frac{\begin{bmatrix} 7 \\ 6 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ 2 \end{bmatrix}}{\begin{bmatrix} 4 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ 2 \end{bmatrix}} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{28 + 12}{16 + 4} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}.$$



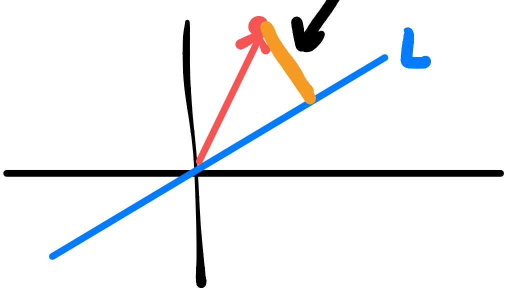
In $\mathbb{R}^2$, the distance from a point (x, y) to a line $L = \mathbb{R}\text{-span}\{u\}$ is the length

$$\left\| \underbrace{\begin{bmatrix} x \\ y \end{bmatrix} - \operatorname{proj}_L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)}_{\in L^{\perp}} \right\|.$$

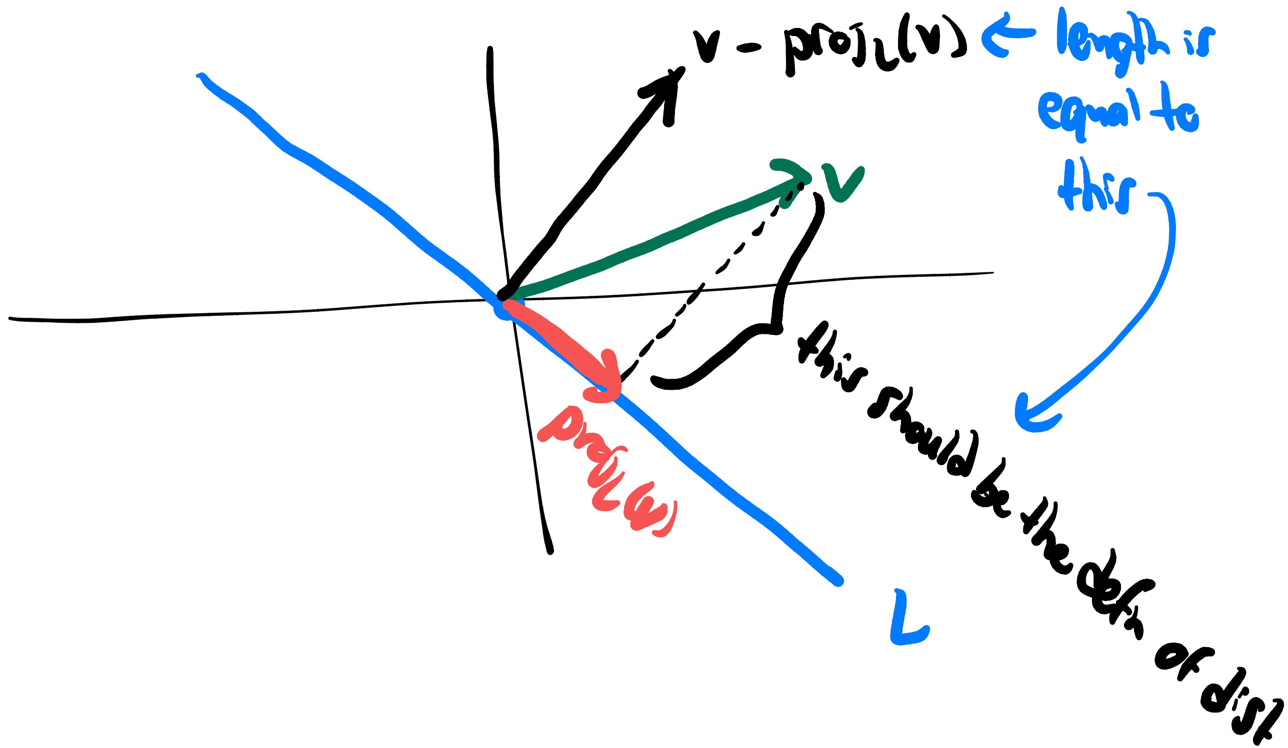
Example. To find the distance from the point $(x, y) = (7, 6)$ to the line L defined by $y = \frac{1}{2}x$, note that L contains the vector $u = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Let $w = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$.

Then $\operatorname{proj}_L\left(\begin{bmatrix} 7 \\ 6 \end{bmatrix}\right) = \frac{w \bullet u}{u \bullet u} u = \frac{28+12}{16+4} u = \frac{40}{20} u = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$ so the distance is

$$\left\| \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\| = \sqrt{1+4} = \sqrt{5}.$$



dist from $\vec{v}$ to L is $\|v - \text{proj}_L(v)\|$:



$$\begin{cases} u_i \cdot u_j = 0 \text{ if } i \neq j \\ \text{AND} \\ u_i \cdot u_i = 1 \text{ for all } i \end{cases}$$

3 Orthonormal vectors

A set of vectors $u_1, u_2, \ldots, u_p$ is *orthonormal* if the vectors are orthogonal and each vector is a unit vector. This means $u_i \bullet u_j = 0$ when $i \neq j$ and $u_i \bullet u_i = 1$ for all i.

An *orthonormal basis* of a subspace is a basis that is orthonormal.

Convention: a square matrix with orthonormal columns is an *orthogonal matrix*.

It would make more sense to call such a matrix an "orthonormal matrix" but the term "orthogonal matrix" is standard and widely used.

Example. The standard basis $e_1, e_2, \ldots, e_n$ is an orthonormal basis for $\mathbb{R}^n$.

Example. $\frac{1}{\sqrt{11}}\begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$, $\frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, and $\frac{1}{\sqrt{66}}\begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix}$ is an orthonormal basis for $\mathbb{R}^3$.

Ex Every rotation matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$ is orthagonal

↓

meaning orthonormal columns

Recall: $(\cos\theta)^2 + (\sin\theta)^2 = 1$

$$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = (\cos\theta)^2 + (\sin\theta)^2 = 1$$

entry (i,j) of $U^T U$ is (inner product of columns i and j)

Theorem. Let U be an $m \times n$ matrix.

The columns of U are orthonormal vectors if and only if $U^\top U = I_n$.

If U is square then its columns are orthonormal if and only if $U^\top = U^{-1}$.

(In other words, a matrix U is *orthogonal* if and only if U is square and $U^\top = U^{-1}$.)

Proof. Suppose $U = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}$ where each $u_i \in \mathbb{R}^m$.

The entry in position (i,j) of $U^\top U$ is then $u_i^\top u_j = u_i \bullet u_j$.

Therefore $u_i \bullet u_i = 1$ and $u_i \bullet u_j = 0$ for all $i \neq j$ if and only if $U^\top U = I_n$. $\square$

Corollary. If U is an orthogonal matrix then $\det(U) \in \{-1, 1\}$.

Proof. We have $\det(U)^2 = \det(U^\top)\det(U) = \det(U^\top U) = \det(I) = 1$. $\square$

$\det(U) = \det(U^T)$

Theorem. Let U be an $m \times n$ matrix with orthonormal columns.

key idea:
multiplying by such a matrix preserves
inner products
lengths
distances
angles

Suppose $x, y \in \mathbb{R}^n$. Then:

1. $\|Ux\| = \|x\|$.
2. $(Ux) \bullet (Uy) = x \bullet y$.
3. $(Ux) \bullet (Uy) = 0$ if and only if $x \bullet y = 0$.

Proof. First and third statements are special cases of second since

$$\|Ux\| = \|x\| \quad \text{if and only if} \quad (Ux) \bullet (Ux) = x \bullet x.$$

The second statement holds since

$$(Ux) \bullet (Uy) = x^\top U^\top U y = x^\top I y = x^\top y = x \bullet y.$$

$\parallel$

$(Ux)^T (Uy) = x^T \cancel{U^T} U y$

$(U^T U = I)$

□

4 Orthogonal projections onto subspaces

We have already seen that if $y \in \mathbb{R}^n$ and $L \subseteq \mathbb{R}^n$ is a 1-dimensional subspace then y can be written uniquely as $y = \widehat{y} + z$ where $\widehat{y} \in L$ and $z \in L^{\perp}$.

This generalizes to arbitrary subspaces as follows:

Words :

Theorem. Let $W \subseteq \mathbb{R}^n$ be any subspace. Let $y \in \mathbb{R}^n$.

Then there are unique vectors $\widehat{y} \in W$ and $z \in W^{\perp}$ such that $y = \widehat{y} + z$.

If $u_1, u_2, \ldots, u_p$ is an orthogonal basis for W then

Formula !

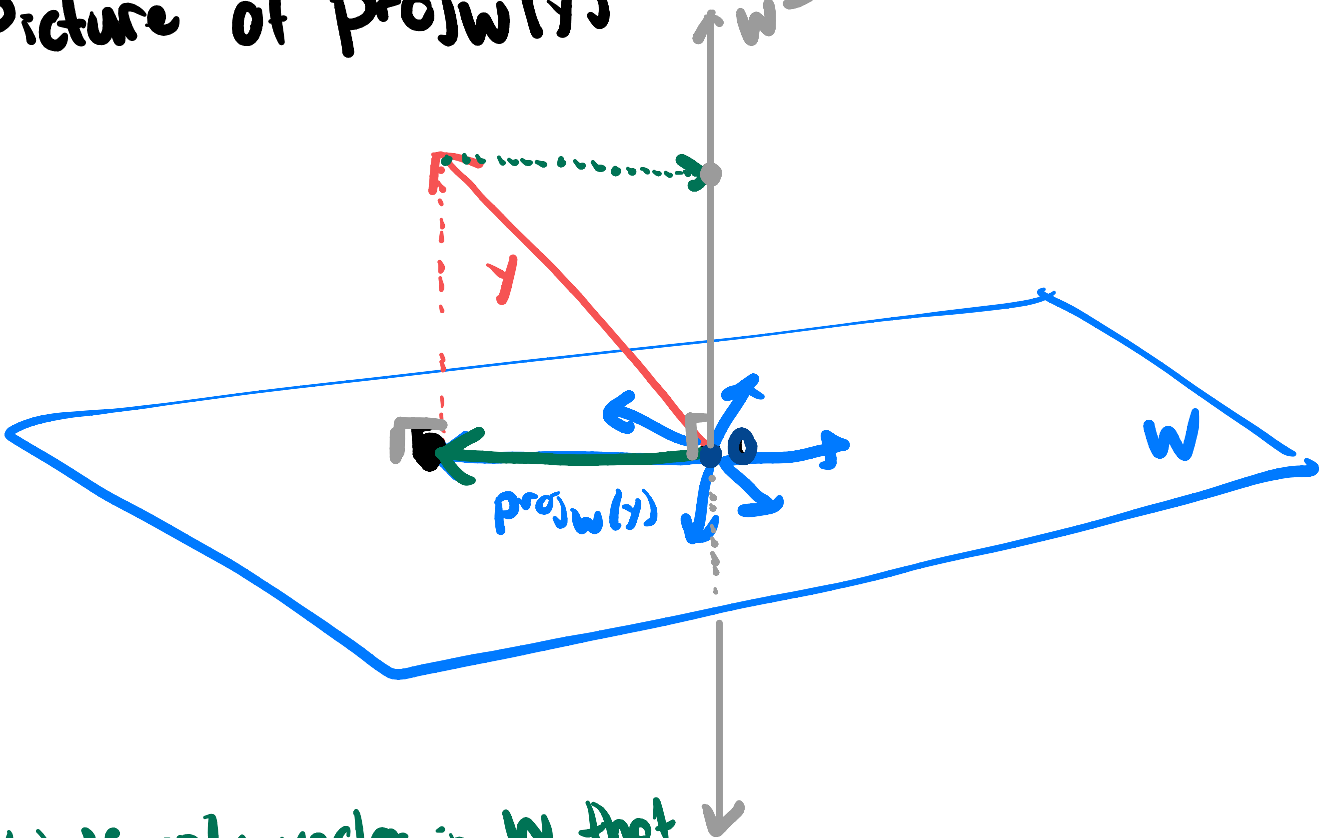
$$\boxed{\widehat{y} = \frac{y \bullet u_1}{u_1 \bullet u_1} u_1 + \frac{y \bullet u_2}{u_2 \bullet u_2} u_2 + \cdots + \frac{y \bullet u_p}{u_p \bullet u_p} u_p} \quad \text{and} \quad \boxed{z = y - \widehat{y}.} \qquad (*)$$

It doesn't matter which orthogonal basis is chosen for W; formula gives same values.

We will call $\widehat{y}$ the orthogonal projection of y onto W

denoted: $\mathrm{proj}_W(y) \in W$

Picture of $proj_W(y)$:



$proj_W(y)$ is only vector in W that points to the spot closest to endpt of y

(dim W = 2)

Definition. The vector $\widehat{y}$, defined relative to y and W by the formula (*) in the preceding theorem, is the *orthogonal projection* of y onto W.

From now on we will write $\boxed{\text{proj}_W(y) = \widehat{y}}$ to refer to this vector.

- If $u_1, u_2, \ldots, u_p$ is an orthogonal basis for W then

$$\boxed{\text{proj}_W(y) = \frac{y \bullet u_1}{u_1 \bullet u_1} u_1 + \frac{y \bullet u_2}{u_2 \bullet u_2} u_2 + \cdots + \frac{y \bullet u_p}{u_p \bullet u_p} u_p.}$$

- If $u_1, u_2, \ldots, u_p$ is an **orthonormal basis** for W then

$$\boxed{\text{proj}_W(y) = (y \bullet u_1)u_1 + (y \bullet u_2)u_2 + \cdots + (y \bullet u_p)u_p.}$$

Note: can convert any orthogonal basis $v_1\ v_2 \cdots v_p$ to be orthonormal: $\frac{1}{\|v_1\|}v_1, \frac{1}{\|v_2\|}v_2, \frac{1}{\|v_3\|}v_3, \ldots$

$\boxed{\textbf{Corollary. } \text{If } W \subseteq \mathbb{R}^n \text{ is any subspace then } \dim W^\perp = n - \dim W.}$

Proof. The preceding theorem shows that W and $W^\perp$ together span $\mathbb{R}^n$.

Therefore the union of any basis for W with a basis for $W^\perp$ also spans $\mathbb{R}^n$.

Hence $\underline{\dim W + \dim W^\perp \geq n}$.

But we showed last time that also $\underline{\dim W^\perp + \dim W \leq n}$. $\square$

any $y \in \mathbb{R}^n$ can be written as

$$y = \underbrace{\text{proj}_W(y)}_{\in W} + \underbrace{(y - \text{proj}_W(y))}_{\in W^\perp}$$

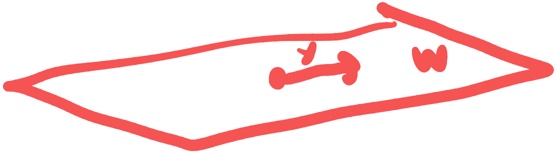
Cor Show that $\operatorname{rank}(A) = \operatorname{rank}(A^T)$

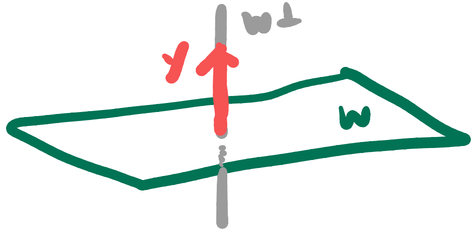
Pf $\operatorname{rank}(A^T) = \dim \operatorname{Col}(A^T) = (\#\text{cols of } A) - \dim \operatorname{Nul}(A^T)$

$$= (\#\text{cols of } A) - \dim \operatorname{Col}(A)^T$$

$$= \cancel{(\#\text{cols of } A)} - (\cancel{n} - \dim \operatorname{Col} A)$$

$$= \operatorname{rank}(A)$$





Properties of orthogonal projections onto a subspace $W \subseteq \mathbb{R}^n$.

Fact. If $y \in W$ then $\text{proj}_W(y) = y$. If $y \in W^\perp$ then $\text{proj}_W(y) = 0$.

Proposition. If $v \in W$ and $y \in \mathbb{R}^n$ and $v \neq \text{proj}_W(y)$ then $\|y - \text{proj}_W(y)\| < \|y - v\|$.

In words: $\boxed{\text{the projection } \text{proj}_W(y) \text{ is the vector in } W \text{ that is closest to } y.}$ ***

alt def.

Proof. Let $\widehat{y} = \text{proj}_W(y)$. Then $y - v = \underbrace{(y - \widehat{y})}_{\in W^\perp} + \underbrace{(\widehat{y} - v)}_{\in W}$.

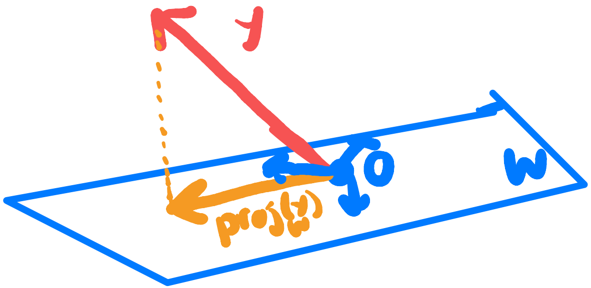
The first term in parentheses is in $W^\perp$ while the second term is in W.

$\widehat{y} \neq v$ so this is > 0

Therefore by the Pythagorean theorem

$$\|y - v\|^2 = \|y - \widehat{y}\|^2 + \|\widehat{y} - v\|^2 > \|y - \widehat{y}\|^2$$

since $\|\widehat{y} - v\| > 0$. $\square$



To compute $proj_W(y)$, we need an orthogonal basis for W

How to find this?

5 The Gram-Schmidt process

The *Gram-Schmidt process* is a list of formulas that turns an arbitrary basis for some subspace of $\mathbb{R}^n$ into an orthogonal basis of the same subspace.

This process gives a constructive proof of the following theorem:

Theorem. Let $W \subseteq \mathbb{R}^n$ be a nonzero subspace. Then W has an orthogonal basis.

(The zero subspace $\{0\}$ has an orthogonal basis given by the empty set, but we exclude this trivial case.)

Gram-Schmidt process. Suppose $x_1, x_2, \dots, x_p$ is any basis for W.

An orthogonal basis is given by the vectors $v_1, v_2, \dots, v_p$ defined inductively by:

orthogonal basis for W

$$\left\{\begin{aligned}
v_1 &= x_1. \\
v_2 &= x_2 - \frac{x_2 \bullet v_1}{v_1 \bullet v_1} v_1. && = x_2 - \mathrm{proj}_{\mathbb{R}\text{-span}(v_1)}(x_2) \\
v_3 &= x_3 - \frac{x_3 \bullet v_1}{v_1 \bullet v_1} v_1 - \frac{x_3 \bullet v_2}{v_2 \bullet v_2} v_2. && = x_3 - \mathrm{proj}_{\mathbb{R}\text{-span}(v_1, v_2)}(x_3) \\
v_4 &= x_4 - \frac{x_4 \bullet v_1}{v_1 \bullet v_1} v_1 - \frac{x_4 \bullet v_2}{v_2 \bullet v_2} v_2 - \frac{x_4 \bullet v_3}{v_3 \bullet v_3} v_3. && = x_4 - \mathrm{proj}_{\mathbb{R}\text{-span}(v_1, v_2, v_3)}(x_4) \\
&\vdots \\
v_p &= x_p - \frac{x_p \bullet v_1}{v_1 \bullet v_1} v_1 - \frac{x_p \bullet v_2}{v_2 \bullet v_2} v_2 - \cdots - \frac{x_p \bullet v_{p-1}}{v_{p-1} \bullet v_{p-1}} v_{p-1}.
\end{aligned}\right.$$

More strongly, we can say the following.

Let $W_i = \mathbb{R}\text{-span}\{v_1, v_2, \dots, v_i\}$ for each $i = 1, 2, \dots, p$.

Then $W_i = \mathbb{R}\text{-span}\{x_1, x_2, \dots, x_i\}$.

Also $v_1, v_2, \dots, v_i$ is an orthogonal basis for W_i and $v_{i+1} = x_{i+1} - \mathrm{proj}_{W_i}(x_{i+1})$.

Remark. To find an **orthonormal** basis for a subspace W:

- First find an orthogonal basis $v_1, v_2, \dots, v_p$.
- Then replace each vector v_i by $\boxed{u_i = \frac{1}{\|v_i\|} v_i.}$

The vectors $u_1, u_2, \dots, u_p$ will then be an orthonormal basis.

Sometimes this extended algorithm is also called the Gram-Schmidt process.

For homework problems that ask you to perform the Gram-Schmidt process, be sure to use the first version which does **not** convert the outputs to unit vectors.

Example. Suppose $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $x_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

These vectors are linearly independent and a basis for $W = \mathbb{R}\text{-span}\{x_1, x_2, x_3\}$.

To compute an orthogonal basis for W, we carry out the Gram-Schmidt process.

- Set $v_1 = x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Then $v_2 = x_2 - \frac{x_2 \bullet v_1}{v_1 \bullet v_1} v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{4} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$.
- Set $v_3 = x_3 - \frac{x_3 \bullet v_1}{v_1 \bullet v_1} v_1 - \frac{x_3 \bullet v_2}{v_2 \bullet v_2} v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$.

Then $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$ are an orthogonal basis for W.

$v_1 \cdot v_2 = 0$ ✓

$v_1 \cdot v_3 = 0$ ✓

$v_2 \cdot v_3 = 0$ ✓